Criticality without FKG

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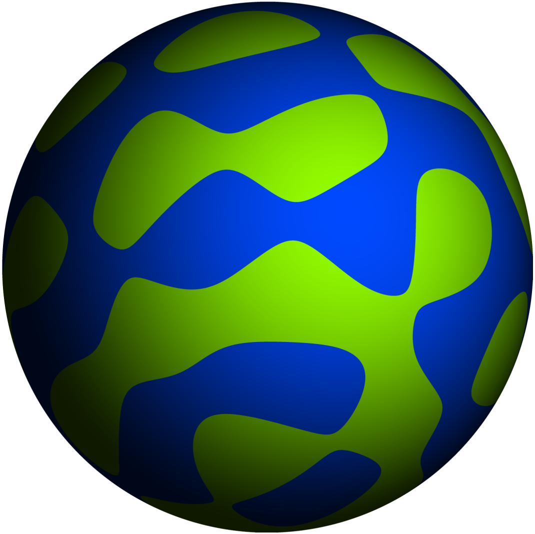
Motivation

Random eigenfunction of the Laplacian on the sphere



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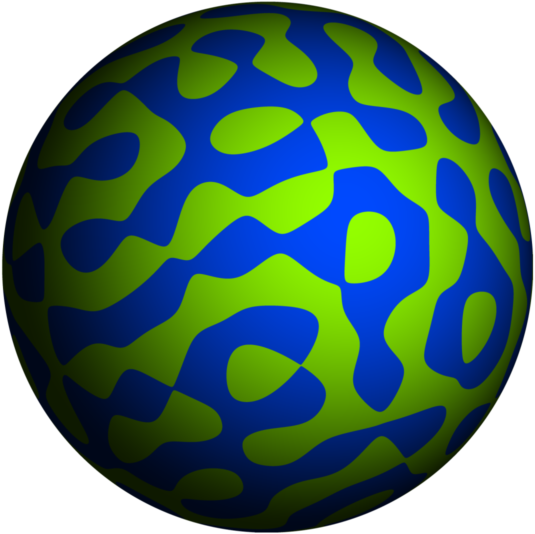

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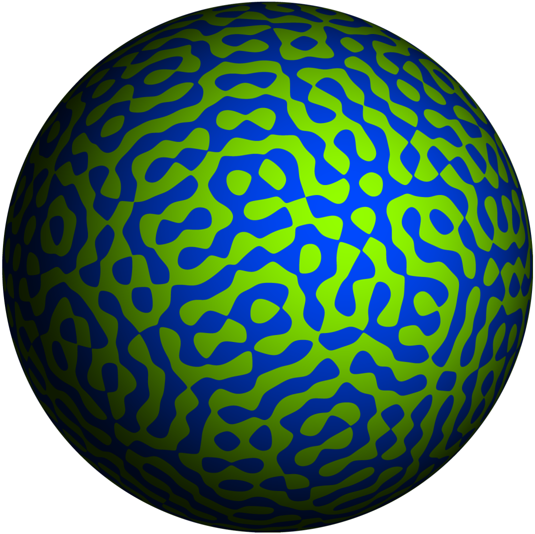

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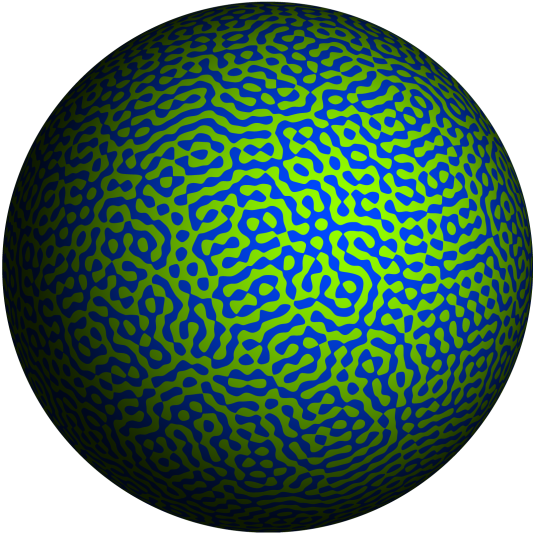

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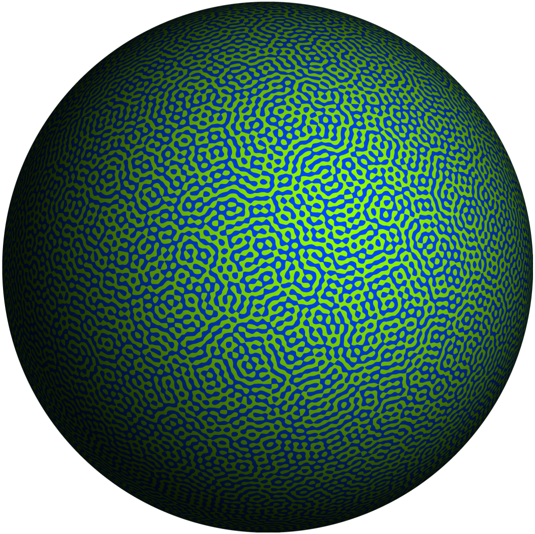

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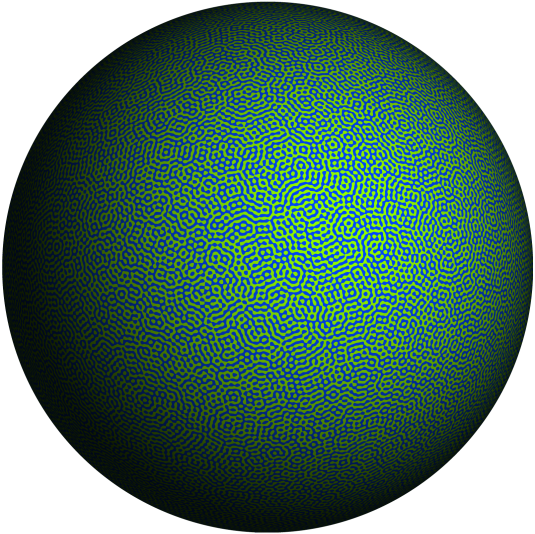

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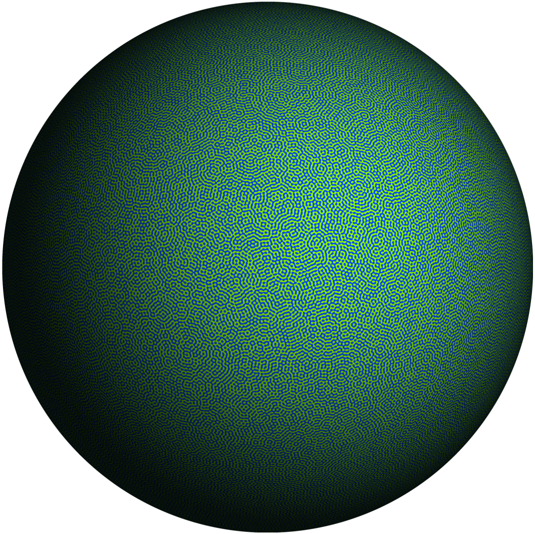


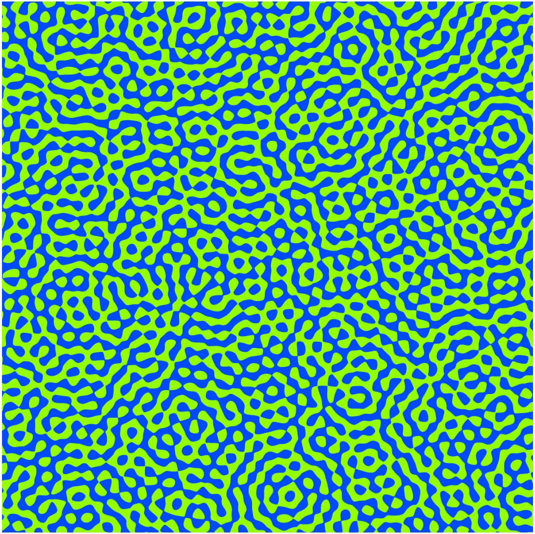
The limit as a Gaussian field

The local limit of random eigenfunctions of Δ as $\lambda \to \infty$ is given by a Gaussian field ϕ of covariance

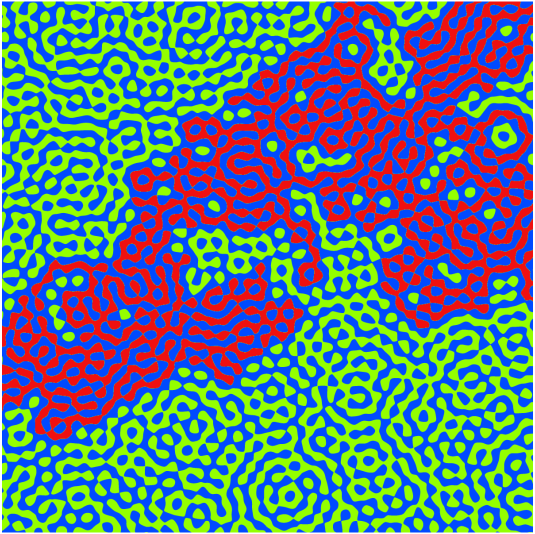
$$Cov[\phi(x), \phi(y)] = J_0(\|y - x\|)$$

The covariance oscillates, and decays as $1/\sqrt{\|y - x\|}$.

Local limit on the sphere



One large connected component



Random polynomial

Define a random homogeneous polynomial on $\mathbb{R}^3$ by

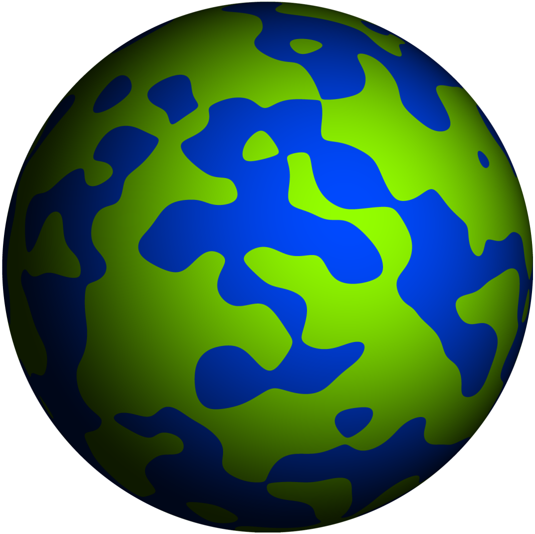
$$P_d(X) = \sum_{|I|=d} a_I \sqrt{\frac{(d+2)!}{I!}} X^I$$

where the a_I are i.i.d. Gaussians.

Restrict it to the unit sphere.

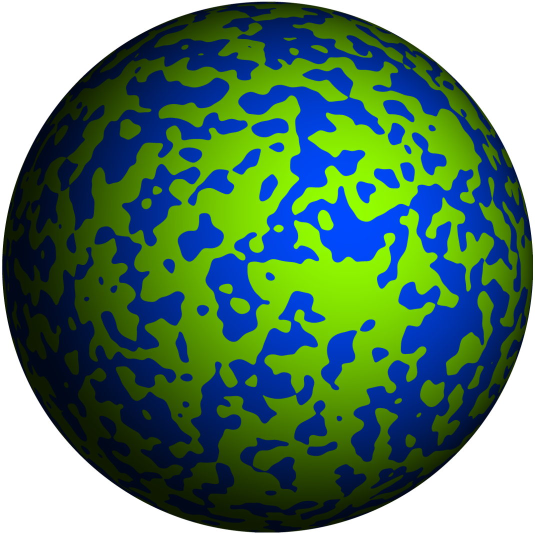
Restriction to the sphere (d=30)

Restriction to the sphere (d=100)

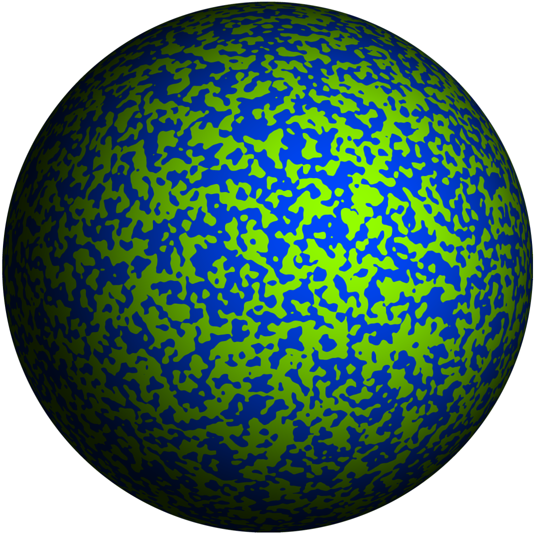


Restriction to the sphere (d=200)

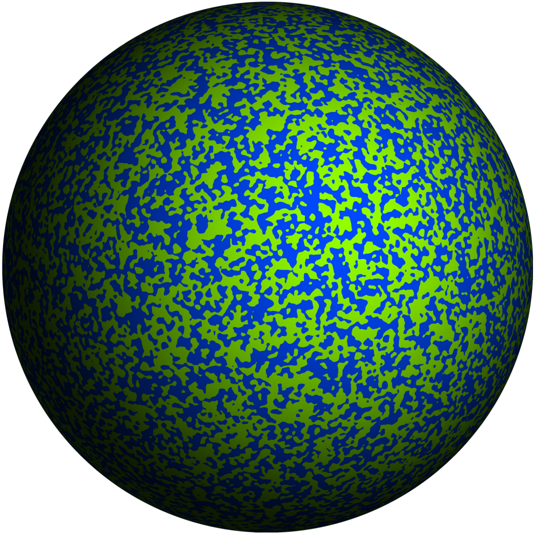
Restriction to the sphere (d=1000)



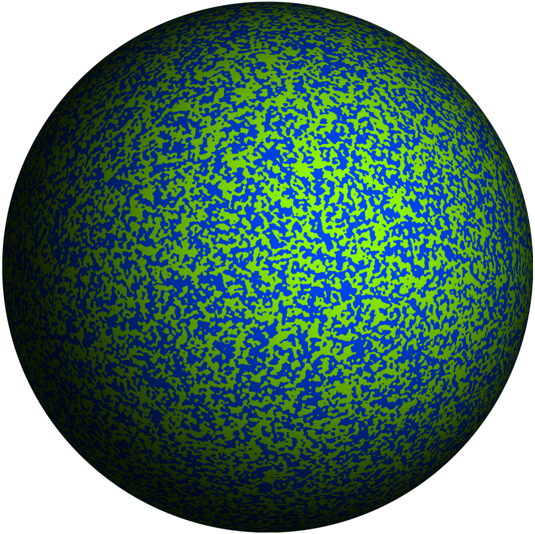
Restriction to the sphere (d=5000)



Restriction to the sphere (d=10000)



Restriction to the sphere (d=20000)



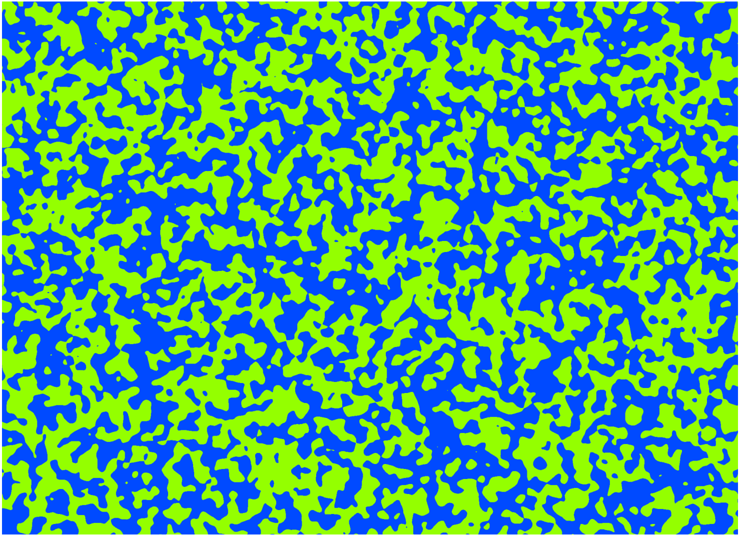
The limit as a Gaussian field

The limit is a stationary centered Gaussian field ψ on $\mathbb{R}^2$, with covariance given by

$$Cov[\psi(x), \psi(y)] = \exp(-\|y - x\|^2/2).$$

In particular, the covariance is positive and decays very fast.

Local limit as $d \to \infty$



The limit as a Gaussian field

$$Q_d(x,y) = \sum_{i+j\leqslant d} a_{ij}\sqrt{\frac{(d+2)!}{i!j!(d-i-j)!}}x^i y^j$$

Rescale by a factor $\sqrt{d}$:

$$Q_d(x/\sqrt{d}, y/\sqrt{d}) \simeq \sum_{i+j\leqslant d} \frac{a_{ij}}{\sqrt{i!j!}}x^i y^j$$

In the limit $d \to \infty$:

$$\psi(x,y) = \sum_{i,j\geqslant 0} \frac{a_{ij}}{\sqrt{i!j!}}x^i y^j$$

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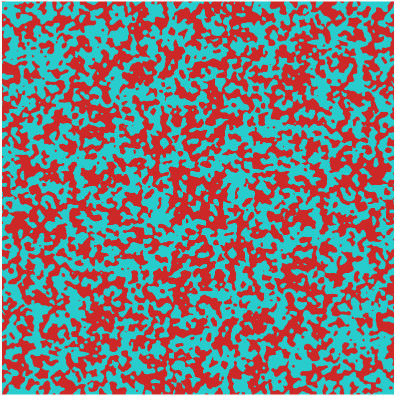
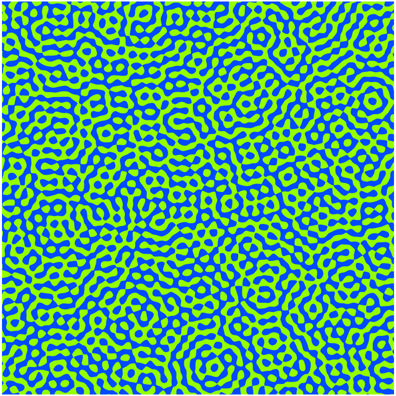
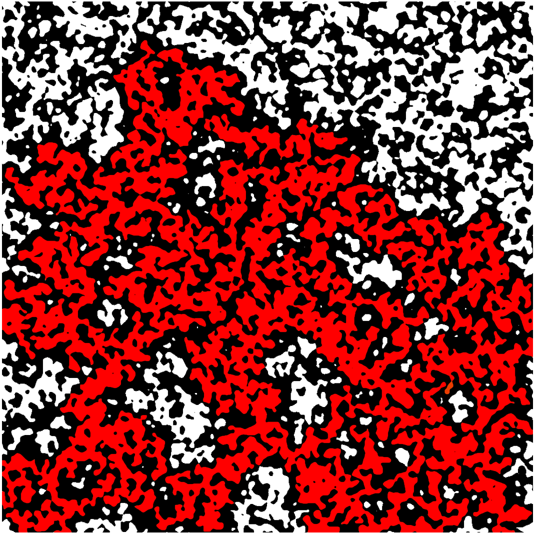
Comparison between the two models

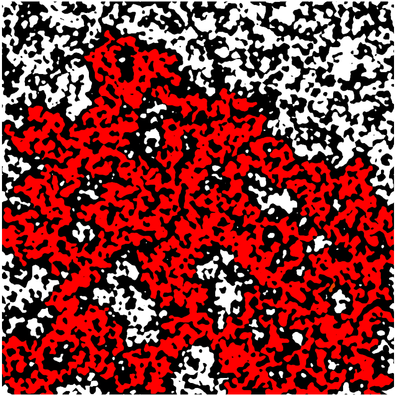


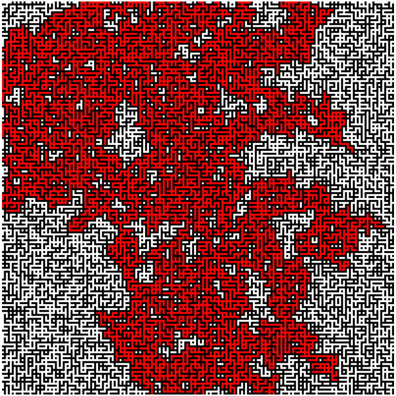


A large connected component in ψ



The same, and a critical percolation cluster

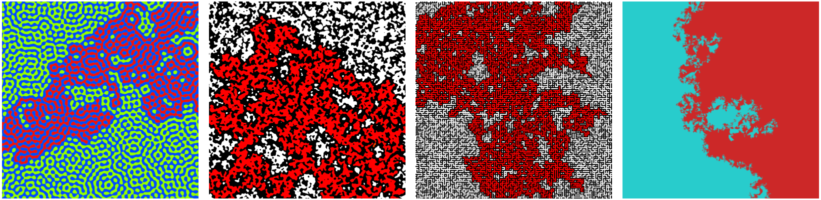




The Bogomolny-Schmidt conjecture

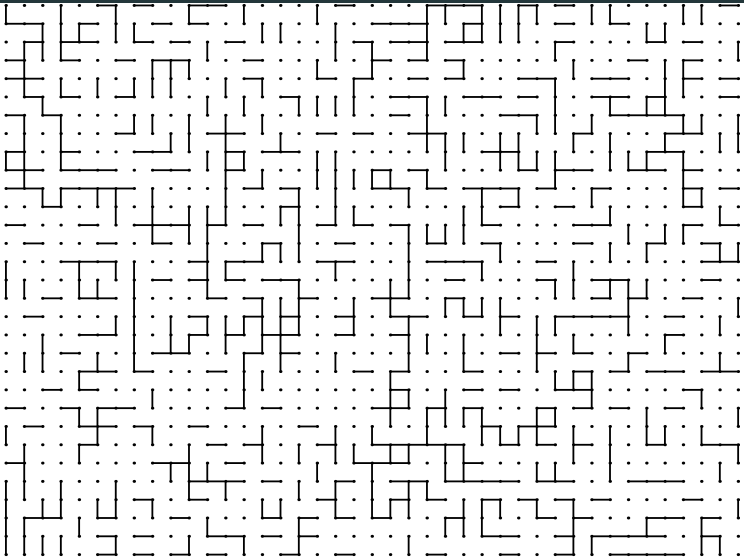
Conjecture

The nodal lines of ϕ (and ψ) converge, in the scaling limit, to the same conformally invariant object as interfaces of critical percolation; in particular, asymptotic crossing probabilities are given by Cardy's formula.

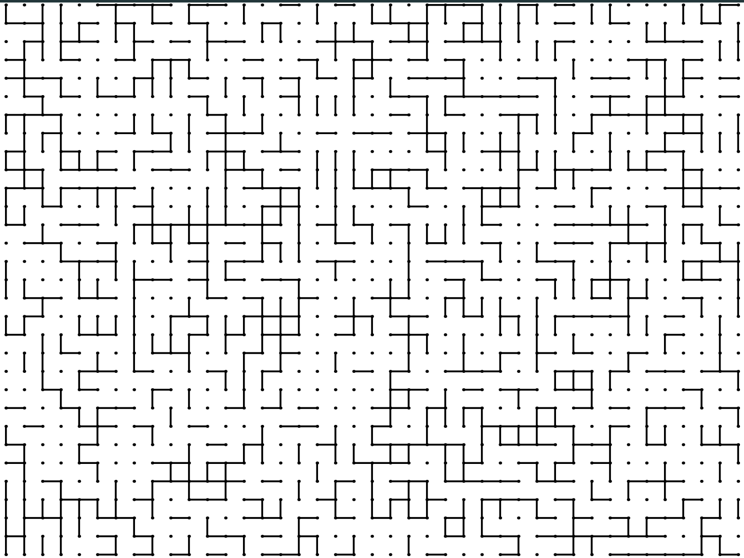


Percolation

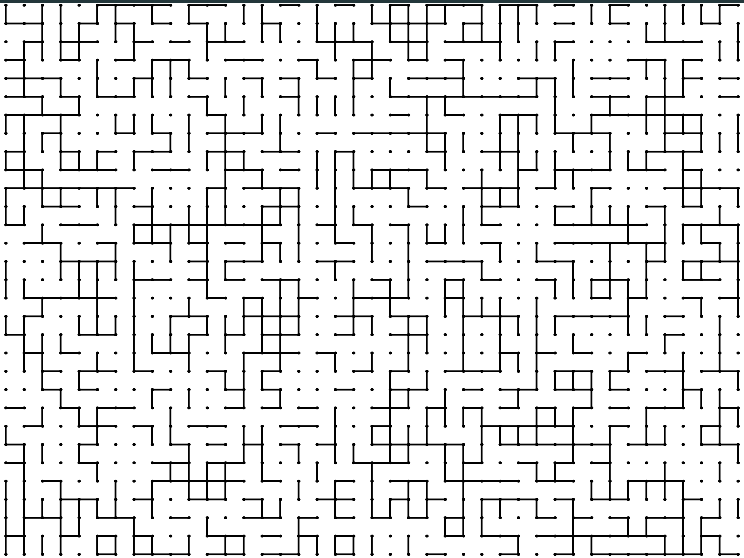
Percolation ($p = 0.3$)



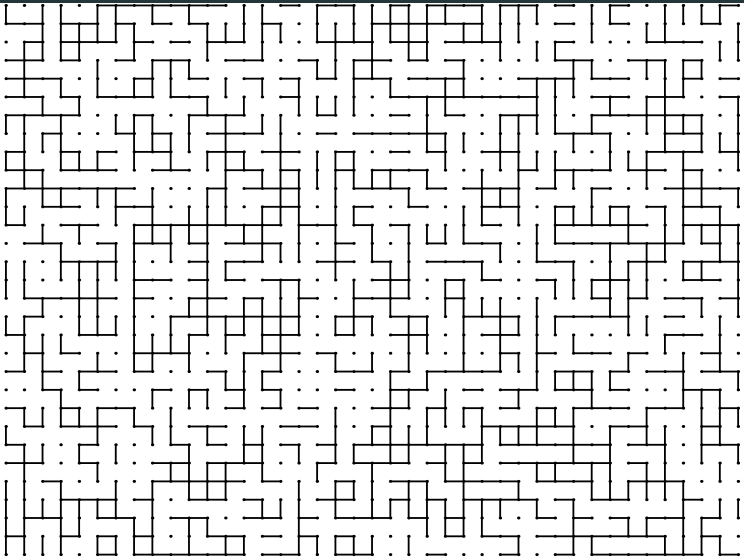
Percolation ($p = 0.4$)



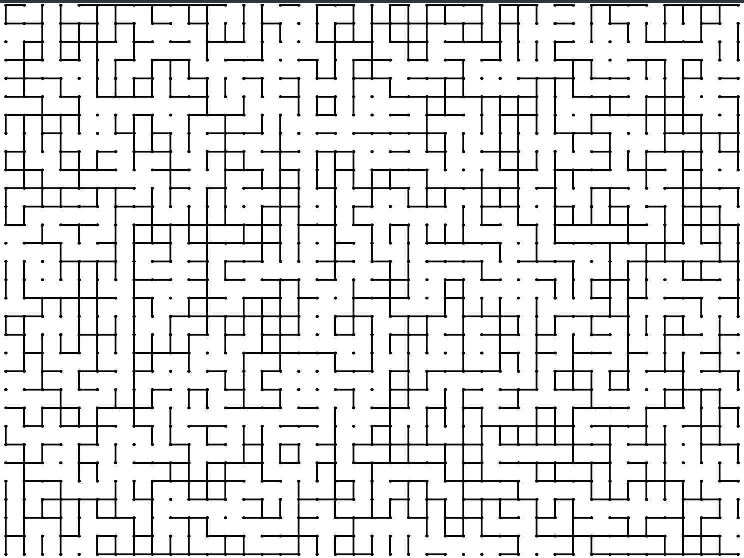
Percolation ($p = 0.45$)



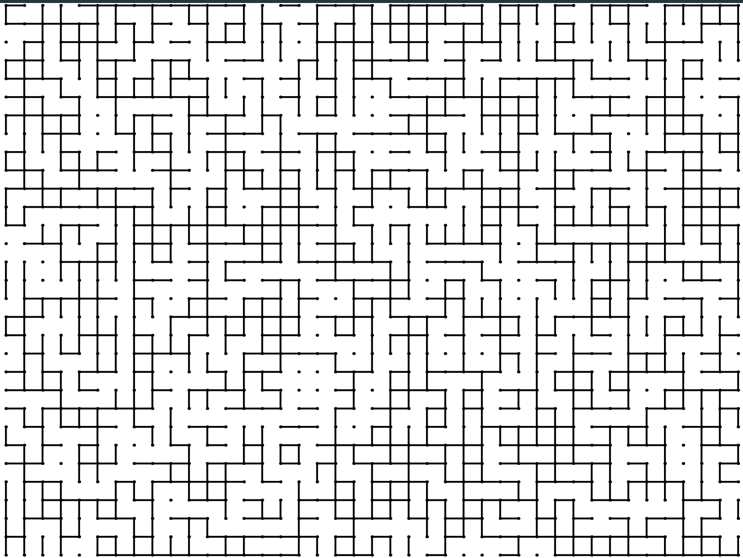
Percolation ($p = 0.5$)



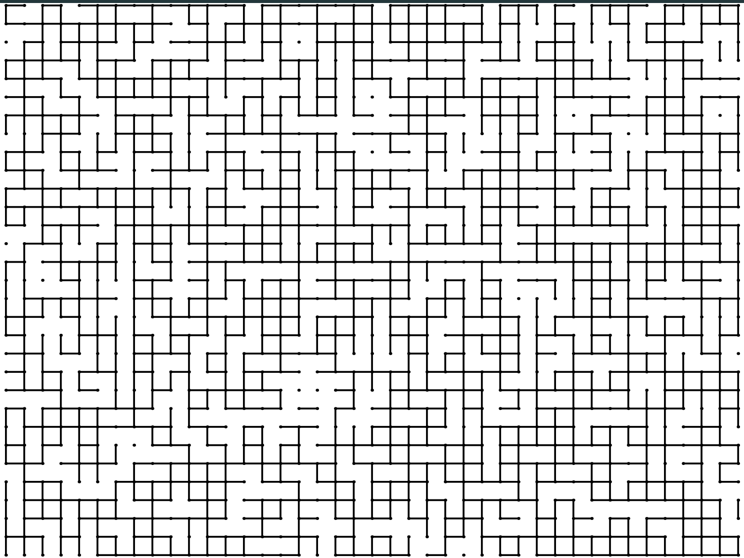
Percolation ($p = 0.55$)



Percolation ($p = 0.6$)



Percolation ($p = 0.7$)



Percolation : classical results

- Kesten (1980) : $p_c = 1/2$
- For $p < p_c$, sub-critical regime :
  - All clusters are a.s. finite
  - $P[0 \longleftrightarrow x] \approx \exp(-\lambda_p \|x\|)$
  - Largest cluster in Λ_n has diameter $\approx \log n$
- For $p > p_c$, super-critical regime :
  - There exists a.s. a unique infinite cluster
  - $P[0 \longleftrightarrow x, |C(x)| < \infty] \approx \exp(-\lambda_p \|x\|)$
  - Largest *finite* cluster in Λ_n has diameter $\approx \log n$
- At $p = p_c$, critical regime :
  - All clusters are a.s. finite
  - $P[0 \longleftrightarrow x] \approx \|x\|^{-5/24}$
  - Largest cluster in Λ_n has diameter $\approx n$

Our aim

We want to show in as much generality as possible that level sets of continuous random fields exhibit the same kind of phase transition at level 0.

To do that: show that the level set of level 0 looks qualitatively much like critical percolation (kind of “weak Bogomolny-Schmidt”) as a first step.

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We want to show in as much generality as possible that level sets of continuous random fields exhibit the same kind of phase transition at level 0.

To do that: show that the level set of level 0 looks qualitatively much like critical percolation (kind of "weak Bogomolny-Schmidt") as a first step.

Note: Muirhead, Rivera, Vanneuville and Köhler-Schindler show that 0 is indeed the critical level under minimal assumptions (in particular, negative correlations are not excluded).

Relevant features of critical percolation (RSW estimates)

Theorem (Box-crossing property (BXP))

For every $\lambda > 0$ there exists $c \in (0,1)$ such that for all n large enough,

$$c \leqslant P_{p_c}[LR(\lambda n, n)] \leqslant 1 - c.$$

Theorem (Arm probability estimates (WB))

There exists $K > 0$ such that for every n,

$$P_{p_c}[0 \longleftrightarrow \partial\Lambda_n] \leqslant Kn^{-1/K},$$

$$P_{p_c}[0 \overset{6}{\longleftrightarrow} \partial\Lambda_n] \leqslant Kn^{-2-1/K}$$

BG2016: get BXP and WB for Bargman-Fock field; aim is to prove them for general fields.

Russo-Seymour-Welsh

Russo-Seymour-Welsh for critical percolation

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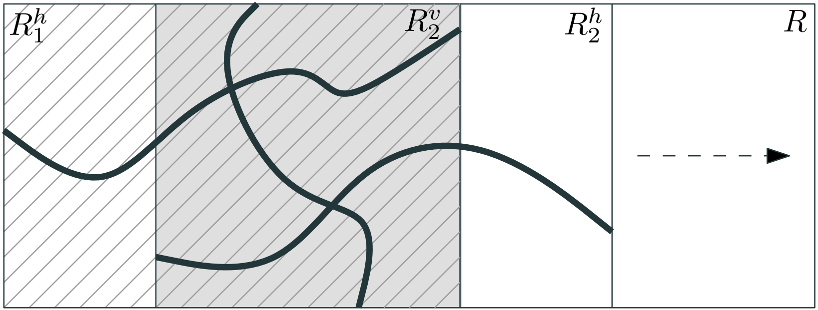
The case $\lambda = 1$ is easy by duality; it is enough to know how the estimate for one value of $\lambda > 1$ and then to glue the pieces.



$$\pi(R) \geqslant \pi(R_1) \times \frac{1}{2} \times \pi(R_2) \times \cdots$$

Russo-Seymour-Welsh: proof ($\lambda = 3/2$)

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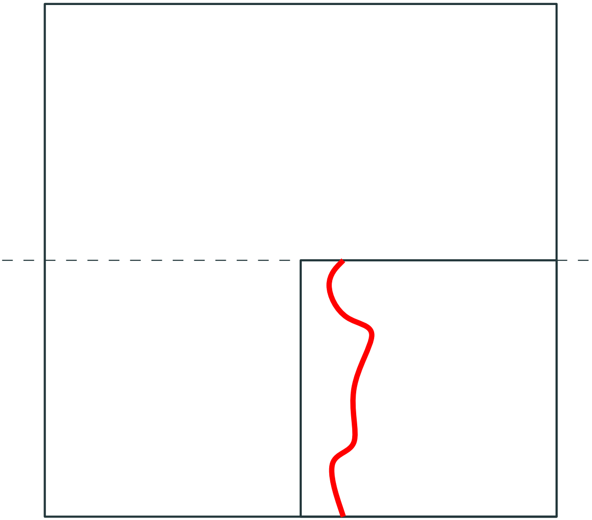

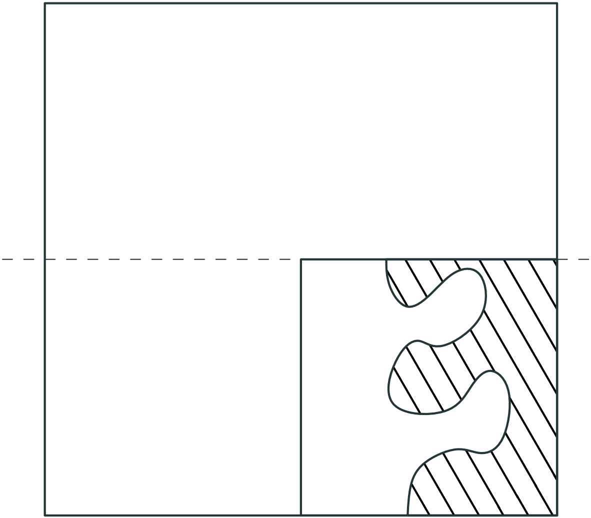

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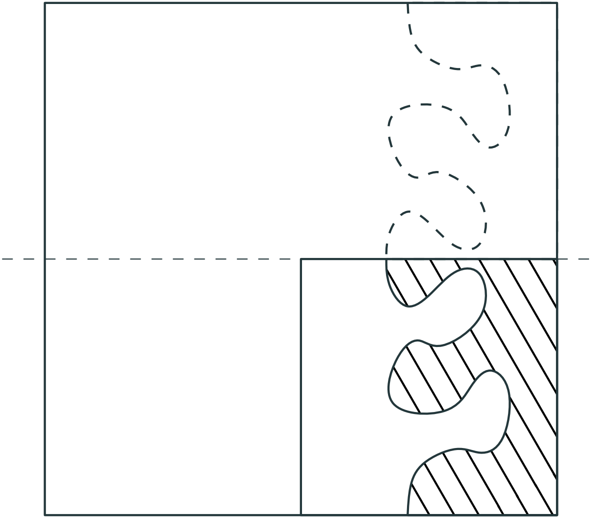

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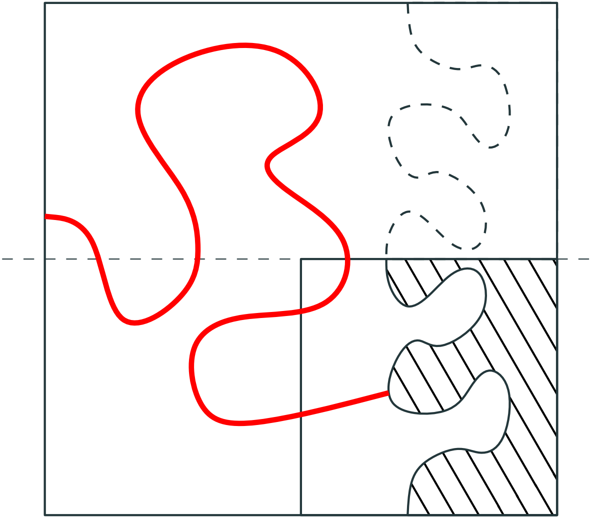

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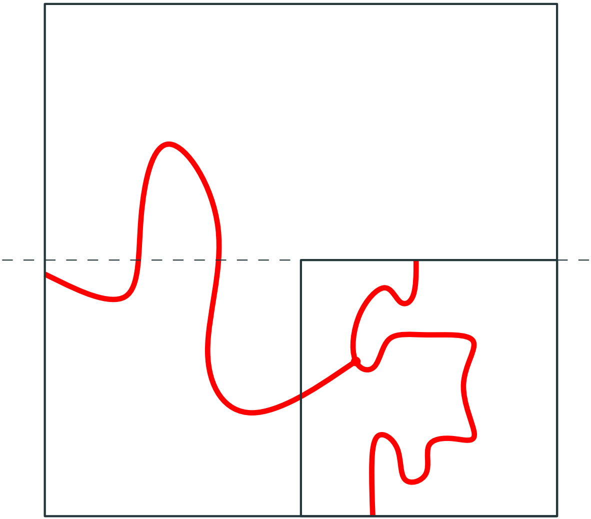

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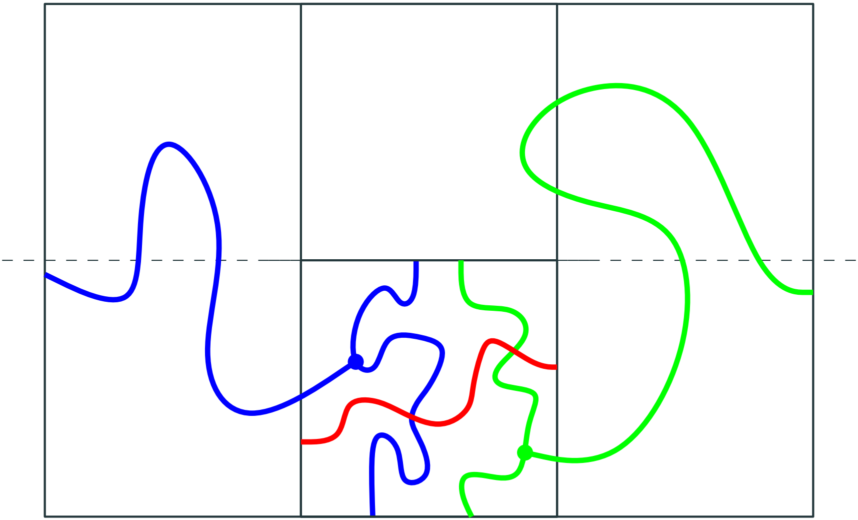

Russo-Seymour-Welsh for the field ψ

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The field ψ satisfies RSW.

A few consequences:

- The set $\{z : \psi(z) > 0\}$ has no unbounded component
- Neither do $\{z : \psi(z) < 0\}$ and $\{z : \psi(z) = 0\}$
- The universal critical exponents are the same as for percolation
- $\psi = 0$ is the critical level [Rivera-Vanneuville]

Negatively correlated discrete fields

Russo-Seymour-Welsh along families of models

Theorem (B., Gayet — WIP)

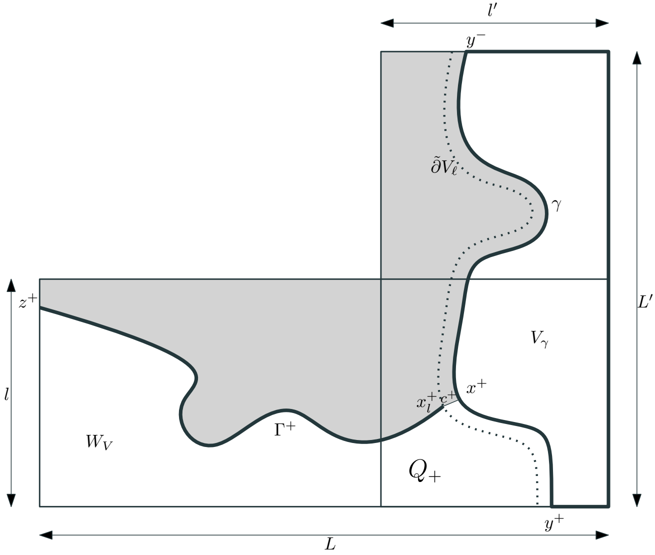
Let (P_u) be a one-parameter family of discrete site models satisfying the following assumptions:

- *symmetry and self-duality;*
- *uniformly good decorrelation;*
- *the Gibbs property;*
- *RSW estimates at parameter $u = 0$.*

Then, RSW estimates hold uniformly for all $u \in (-\varepsilon, \varepsilon)$.

This applies in particular to Ising with possibly negative β.

Key step in the proof



Key step in the proof

Lemma (sketch)

$$\pi(\mathcal{L}) \geqslant \pi(\mathcal{R}_1)\pi(\mathcal{R}_2) - 3\theta(\ell, L) - \beta(\ell, L)$$

with the following quantities in the error term:

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Then, assume RSW at one scale, estimate β at the next scale, and use the estimate to obtain RSW at the next scale.

More detail on the error term β

- A *quad* is a simply connected region in the plane with two marked intervals along its boundary; it is *crossed* by a configuration if these two intervals are connected by an open path contained in the quad;

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- $\mathcal{Q}(r, R, L, L')$ is the collection of all "stochastic" algorithms on $\Lambda_{L'}$ returning either a quad in $Q(r, R, L)$ containing no revealed bit, or $\varnothing$; these are maps

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- The *bad set* $B(\Phi)$ of an algorithm $\Phi \in \mathcal{Q}(r, R, L, L')$ is the collection of all configurations ω on $\Lambda_{L'}$ for which $\Phi(\omega)$ is (nonempty and) crossed

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$$\forall r, R < L, \quad \beta_P(r, R, L) \leqslant \eta^{-1}(r/R)^\eta$$

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- Main scheme: for large L, (L, η)-good $\implies$ (L^{1+c}, η)-good