An entropy proof of the Erdős-Kleitman-Rothschild theorem

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Definition. A graph G is **H-free** if $H \nsubseteq G$

not necessarily induced subgraph

$$\mathcal{F}_n(H) := \{ G \subseteq K_n : H \nsubseteq G \}$$

complete graph with vx set $\{1, \ldots, n\}$

Turán's problem. Determine

$$\mathrm{ex}(n, H) := \max \{ e(G) : G \in \mathcal{F}_n(H) \}$$

This talk. What does a typical H-free graph look like?

Theorem (EKR 1976) Draw $G_n \in \mathcal{F}_n(K_3)$ uniformly at random. Then

$$\lim_{n\to\infty} \mathbb{P}(G_n \text{ is bipartite}) = 1.$$

This is a counting statement: $|\{G \in \mathcal{F}_n(K_3) : \chi(G) > 2\}| \ll |\{G \subseteq K_n : \chi(G) \leq 2\}|$.

Moreover, for every fixed $r \geq 3$,

$$2^{ex(n,K_r)} \leq \mathcal{F}_n(K_r) \leq 2^{ex(n,K_r)+o(n^2)}.$$

trivial

Theorem (Erdös-Frankl-Rödl 1986) For every H,

$$|\mathcal{F}_n(H)| = 2^{ex(n,H)+o(n^2)}$$

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Proof methods:

EKR: Kővári-Sós-Turán / Erdős + induction on n & r

EFR: regularity lemma

BMS+ST: hypergraph containers + supersaturated Turán's theorem

KMPS: Shannon's entropy + supersaturated Turán's theorem

Entropy. X: random var. taking values in a finite set $\mathcal{X}$.

$$H(X) := -\sum_{x \in \mathcal{X}} \mathbb{P}(X=x) \log_2 \mathbb{P}(X=x)$$

chosen for convenience today

Informally, $H(X) = \mathbb{E}[\text{surprise}(X)]$.

Conditional entropy. X, Y: random vars taking values in finite sets $\mathcal{X}, \mathcal{Y}$, respectively.

X conditioned on $\{Y=y\}$

$$H(X|Y) := \mathbb{E}[H(X_{\{Y=y\}})]$$

$$= -\sum_{y \in \mathcal{Y}} \mathbb{P}(Y=y) \sum_{x \in \mathcal{X}} \mathbb{P}(X=x|Y=y) \log_2 \mathbb{P}(X=x|Y=y)$$

Properties of entropy:

① $0 \leq H(X) \leq \log_2 |\mathcal{X}|$

'=' iff X is uniform on $\mathcal{X}$

'=' iff X and Y are independent

② $H(X|Y) = H(X,Y) - H(Y) \leq H(X)$

Corollary: $H(X_1, \dots, X_n | Y) \leq \sum_{i=1}^{n} H(X_i | Y)$

③ $H(X|Y,Z) \leq H(X|Y)$

④ Pinsker's inequality:

$$d_{TV}\big((X,Y), X \times Y\big) \leq \sqrt{2} \cdot \sqrt{H(X) - H(X|Y)}$$

total variation distance

joint dist.

random vector with independent coordinates

An entropy proof of: $\log_2 |\mathcal{F}_n(K_3)| = ex(n, K_3) + o(n^2)$.

Choose $X \in \mathcal{F}_n(K_3)$ u.a.r. so that $H(X) \overset{①}{=} \log_2 |\mathcal{F}_n(K_3)|$.

Think of X as the vector $(X_e)_{e \in K_n}$.

Key Lemma. There exists $F \subseteq K_n$ with $e(F) \leq n^{2-\varepsilon}$ s.t., letting $p_e := \mathbb{E}[X_e \mid (X_f)_{f \in F}]$, we have

$(X_f)_F$-measurable r.v.

$$\sum_{ijk} \mathbb{E}[p_{ij} \cdot p_{ik} \cdot p_{jk}] \leq n^{3-5\varepsilon}.$$

When $F = \emptyset$, then $p_e \approx \frac{1}{4}$ for each $e \in K_n$.

When $F = K_n$, then $p_{ij} \cdot p_{ik} \cdot p_{jk} = 0$ for all i,j,k.

Define $G := \{ e \in K_n : p_e \geq n^{-\varepsilon} \}$. $(X_f)_F$-measurable random graph

$$N_{K_3}(G) \cdot n^{-3\varepsilon} \leq \sum_{i,j,k} p_{ij} p_{ik} p_{jk}$$

Key Lemma $\Downarrow$ $F \subseteq K_n$

$(X_f)_F$-measurable event

Let $B := \{ \sum_{i,j,k} p_{ij} p_{ik} p_{jk} > n^{3-4\varepsilon} \}$ and note that

$$B^c \Rightarrow N_{K_3}(G) \leq n^{3-\varepsilon} \Longrightarrow \underline{e(G) \leq ex(n, K_3) + O(n^{2-\delta})}.$$

Erdős-Simonovits ('supersaturation' — a simple 'averaging' argument)

Key Lemma + Markov's ineq. $\Rightarrow$ $\underline{\mathbb{P}(B) \leq n^{-\varepsilon}}$

For every $F' \subseteq F$, let $X_e^{F'} := X_e|_{(X_f)_F = F'}$ $\quad p_e(F') = \mathbb{E}[X_e^{F'}]$

$$H(X) \overset{②}{=} H((X_f)_F) + \underline{H((X_e)_{K_n \setminus F} \mid (X_f)_F)}$$
$$\leq H((X_f)_F) + \mathbb{P}(\mathcal{B}) \cdot \max\{H((X_e^{F'})_{K_n \setminus F}) : F' \in \mathcal{B}\}$$
$$+ \mathbb{P}(\mathcal{B}^c) \cdot \underline{\max\{H((X_e^{F'})_{K_n \setminus F}) : F' \notin \mathcal{B}\}}$$
$$\overset{①}{\leq} e_F + \mathbb{P}(\mathcal{B}) \cdot e(K_n \setminus F)$$
$$+ \mathbb{P}(\mathcal{B}^c) \cdot \max\{\textstyle\sum_{e \in K_n \setminus F} H(X_e^{F'}) : F' \notin \mathcal{B}\}$$
$$\overset{KL}{\leq} \underbrace{n^{2-\varepsilon} + n^{-\varepsilon} \cdot \binom{n}{2}}_{\leq 2n^{2-\varepsilon}} + \underbrace{\max\{\textstyle\sum_{e \in K_n} H(X_e^{F'}) : F' \notin \mathcal{B}\}}_{\text{main term}}$$

$$\log_2 |\mathcal{F}_n(K_3)| \leq \max\{\textstyle\sum_{e\in K_n} H(X_e^{F'}) : F' \notin \mathcal{B}\} + 2n^{2-\varepsilon}$$

- $X_e^{F'} = X_e|_{(X_f)_F = F'}$
- $G^{F'} = \{e \in K_n : \mathbb{E}[X_e^{F'}] = p_e(F') \geq n^{-\varepsilon}\}$
- $p \mapsto H(\mathrm{Ber}(p)) \nearrow$ on $[0, 1/2]$

Finally,

$$\begin{aligned}\sum_{e\in K_n} H(X_e^{F'}) &= \sum_{e\in K_n} H(\mathrm{Ber}(p_e)) \\ &\leq e(G^{F'}) + e(K_n \setminus G^{F'}) \cdot \max_{p \leq n^{-\varepsilon}} H(\mathrm{Ber}(p)) \\ &\leq e(G^{F'}) + O(n^{2-\varepsilon}\log n)\end{aligned}$$

and

$N_{K_3}(G') \leq n^{3-\varepsilon}$ supersaturation

$$\max\{e(G^{F'}) : F' \notin \mathcal{B}\} \leq \mathrm{ex}(n, K_3) + O(n^{2-\delta}).$$

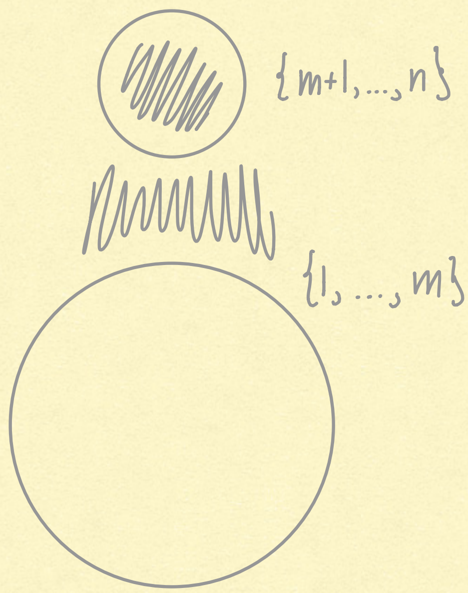
Key Lemma. There exists $F \subseteq K_n$ with $e(F) \leq n^{2-\varepsilon}$ s.t., letting $p_e := \mathbb{E}[X_e \mid (X_f)_{f \in F}]$, we have

$(X_f)_F$ - measurable r.v.

$$\sum_{i,j,k} \mathbb{E}[p_{ij} \cdot p_{ik} \cdot p_{jk}] \leq n^{3-5\varepsilon}.$$

Proof:

Define $F_m \ni ij \iff \max\{i,j\} > m$ s.t.



- $\emptyset = F_n \subseteq F_{n-1} \subseteq \cdots \subseteq F_1 = K_n$
- $e(F_m) \leq n(n-m)$
- $F_{m-1} \setminus F_m \ni ij \iff \max\{i,j\} = m$

Let

$$h_m := H(X_{12} \mid (X_f)_{F_m}).$$

$0 \le H(X) \le \log_2 |\mathcal{X}|$ $H(X \mid Y, Z) \le H(X \mid Y)$

By ① & ③, since $F_m \subseteq F_{m-1}$,

$$1 \geq h_n \geq h_{n-1} \geq \cdots \geq h_1 \geq 0.$$

In particular, there must be $m \in \{n - \sqrt{n}, \dots, n\}$ s.t.

$$h_m - h_{m-1} \leq 1/\sqrt{n}.$$

We set $F := F_m$.

$$\underbrace{H(X_{12} \mid (X_f)_F)}_{h_m} - \underbrace{H(X_{12} \mid (X_f)_{F_{m-1}})}_{h_{m-1}} \leq 1/\sqrt{n}$$

$$\Big\Downarrow \; \text{③} \quad F \setminus F_{m-1} = F_m \setminus F_{m-1} \ni \{1,m\}, \{2,m\}$$

$$\overbrace{H(X_{12} \mid (X_f)_F)}^{\substack{h_m \\ \parallel}} - \overbrace{H(X_{12} \mid (X_f)_F, X_{1m}, X_{2m})}^{\geq h_{m-1}} \leq 1/\sqrt{n}$$

$$\Big\Updownarrow \quad F' \sim (X_f)_F$$

$$\mathbb{E}\Big[\underbrace{H(X_{12}^{F'}) - H(X_{12}^{F'} \mid X_{1m}^{F'}, X_{2m}^{F'})}_{\circledast}\Big] \leq 1/\sqrt{n}$$

$$\Big\Downarrow \quad d_{TV}((X,Y), X \times Y) \leq \sqrt{2} \cdot \sqrt{H(X) - H(X|Y)}$$

$$\mathbb{E}\Big[d_{TV}\big((X_{12}^{F'}, X_{1m}^{F'}, X_{2m}^{F'}), X_{12}^{F'} \otimes (X_{1m}^{F'}, X_{2m}^{F'})\big)\Big] \overset{\text{Pinsker}}{\leq} \mathbb{E}\Big[\sqrt{2 \circledast}\Big] \overset{\text{Jensen}}{\leq} \sqrt{2}\, n^{-1/4}$$

$$\mathbb{E}\left[d_{TV}\left(\left(X^{F'}_{12}, X^{F'}_{1m}, X^{F'}_{2m}\right), X^{F'}_{12} \times \left(X^{F'}_{1m}, X^{F'}_{2m}\right)\right)\right] \le \sqrt{2}\, n^{-1/4}$$

By symmetry (every permutation of $\{1,\dots,m\}$ fixes F), we may replace $(1,2,m)$ with any $(i,j,k) \in \{1,\dots,m\}^3$.

$$\mathbb{E}[p_{ij} \cdot p_{ik} \cdot p_{jk}] \le \mathbb{E}\Big[d_{TV}\Big(\underbrace{\left(X^{F'}_{ij}, X^{F'}_{ik}, X^{F'}_{jk}\right)}_{\mathbb{P}_1}, \underbrace{X^{F'}_{ij} \times X^{F'}_{ik} \times X^{F'}_{jk}}_{\mathbb{P}_2}\Big)\Big] \overset{\Delta\text{-ineq.}}{\le} 2\sqrt{2}\, n^{-1/4}$$

consider the event $T := \{X_{ij} X_{ik} X_{jk} = 1\}$:
$\mathbb{P}_1(T) = 0$ & $\mathbb{P}_2(T) = p_{ij} \cdot p_{ik} \cdot p_{jk}$

If $\max\{i,j,k\} > m$, then $p_{ij} p_{ik} p_{jk} = 0$.

$\sum_{i,j,k} \cdots \blacksquare$