Good News and Bad News for Tree Reconstruction

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(Oxford)

MSI Colloquium 28/04/2021

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multiset!

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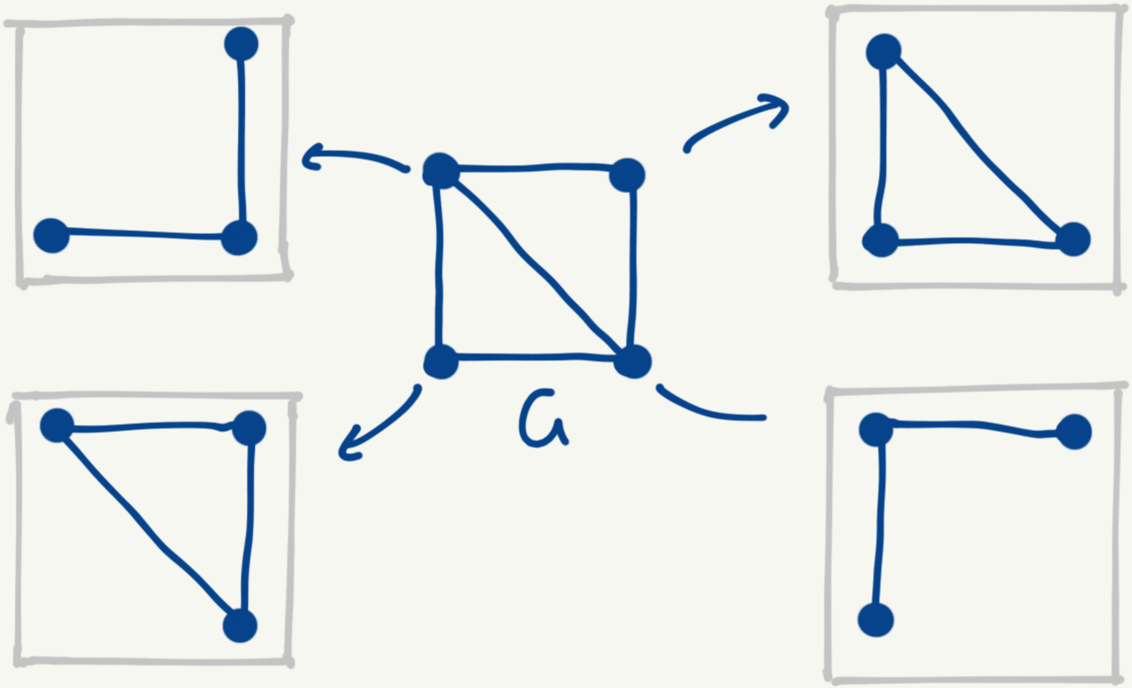
e.g



So $D(\boxtimes) = \{ \underset{P_3}{\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet}, \underset{P_3}{\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet}, \underset{C_3}{\triangle}, \underset{C_3}{\triangle} \}$

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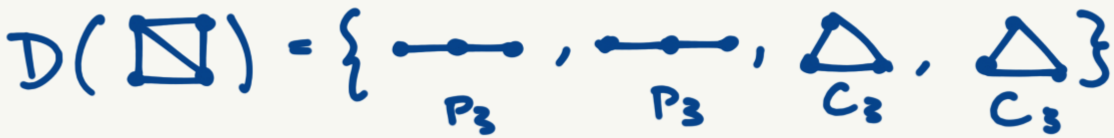

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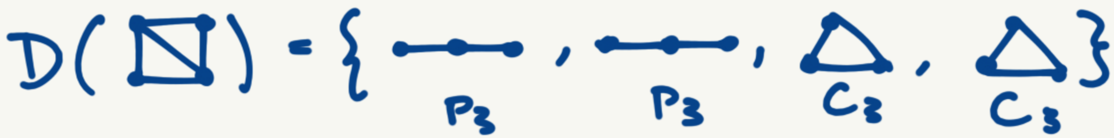

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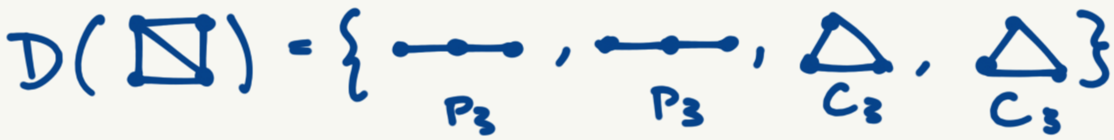

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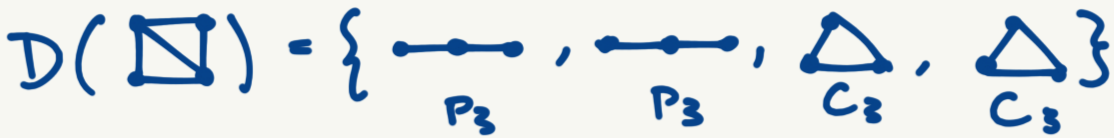

- Elements of the deck are called **cards**, and they come unlabelled.
- A graph is **reconstructible** if it is uniquely determined by its deck.

Q: Which graphs are reconstructible?

$$D(\bullet\!\!-\!\!\!-\!\!\bullet) = \{\bullet\,,\,\bullet\} = D(\bullet\ \ \bullet)$$

so $\bullet\!\!-\!\!\bullet$ and $\bullet\ \ \bullet$ are <u>not</u> reconstructible.

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Every graph on ≥ 3 vertices is reconstructible.

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• # vertices	• Regular graphs
• # edges	• Disconnected graphs
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• Planarity	• Maximal planar
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e.g $D(G) = \{$ , , ,  $\}$

Then # vertices $= 3+1 = 4$

edges $= \dfrac{8}{4-2} = 4$

Degree seq $= 2,2,2,2$

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so G =  $\cong C_4$.

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We have so far discussed classical vertex reconstruction.

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There is also:

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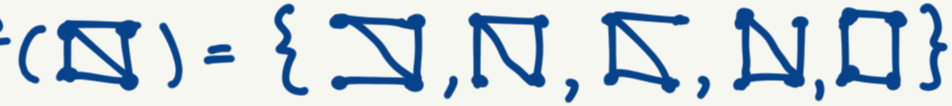
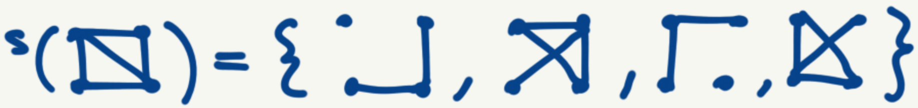
- Edge reconstruction D^e 
- Switching reconstruction D^s 

- Set reconstruction $D^{set}(G) = \{G - v : v \in V(G)\}$ → set!
- Reconstruction from an incomplete deck
- Reconstruction with missing cards
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Observe:
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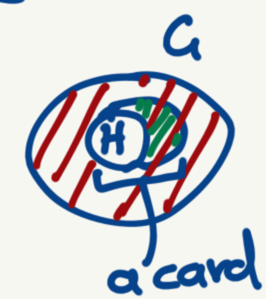
<u>Pf</u> $$\#\text{ copies } H \text{ in } G = \frac{\sum_{C \in D_\ell(G)} \#\text{ copies } H \text{ in } C}{\binom{n-|V(H)|}{\ell-|V(H)|}}$$



$\square$

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Theorem (Kostochka, Nahvi, West, Zirlin 2021++)
Trees with ≥ 22 vertices are reconstructible from the $(n-3)$-deck.

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GOOD NEWS **Theorem** (2021+) For $n \geq 3$, any n-vertex tree is reconstructible from the ℓ-deck whenever $\ell > \frac{8}{9}n + \frac{4}{9}\sqrt{8n+5} + 1$.

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Weak reconstruction

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High diam

Extension counting

Low diam

Greenwell-Hemminger max subgraph counting

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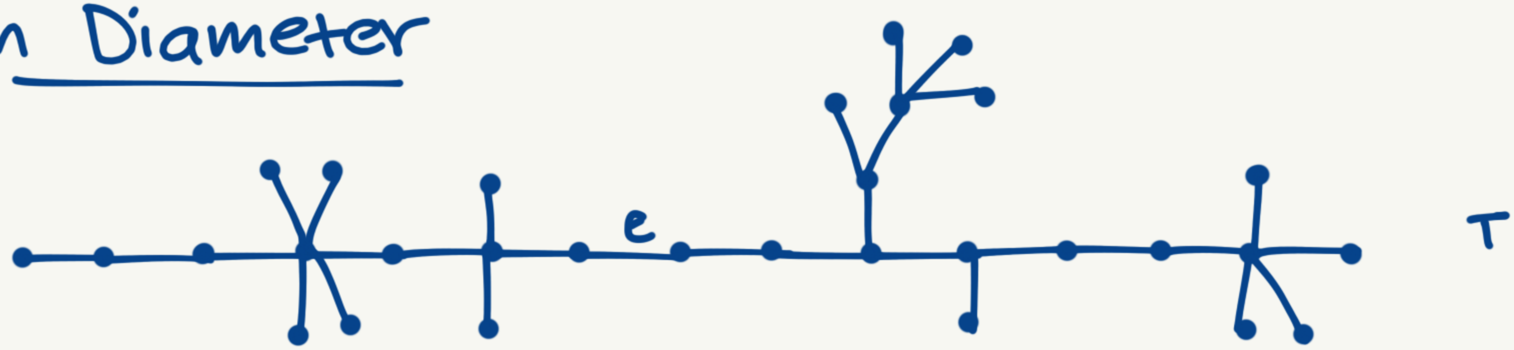

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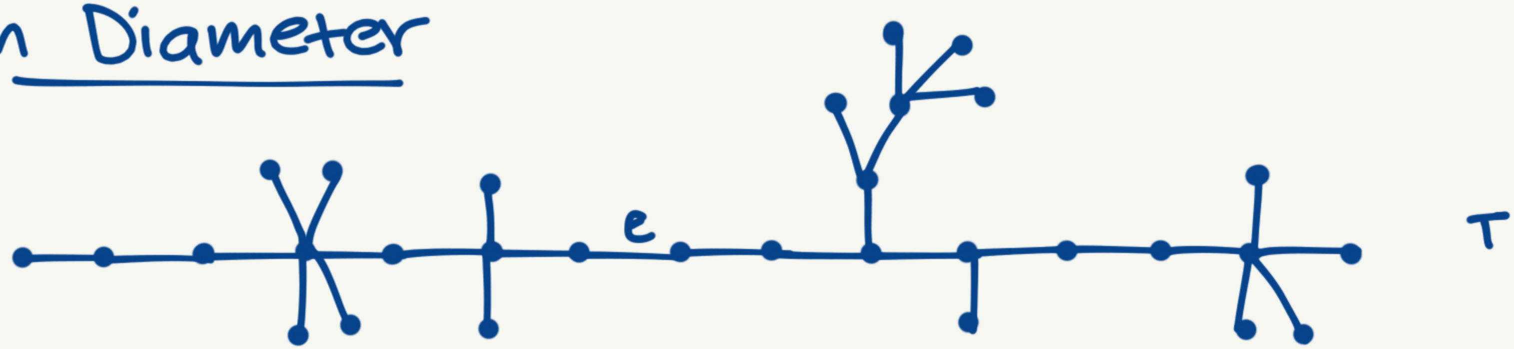

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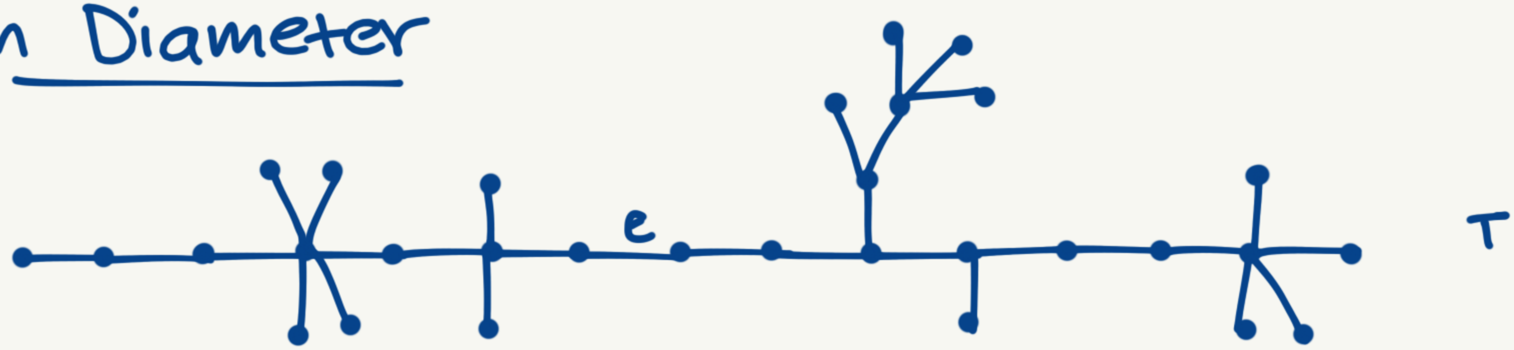

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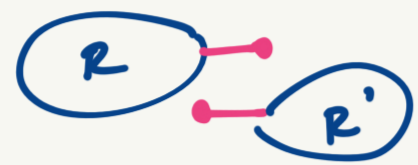

good leaf extension

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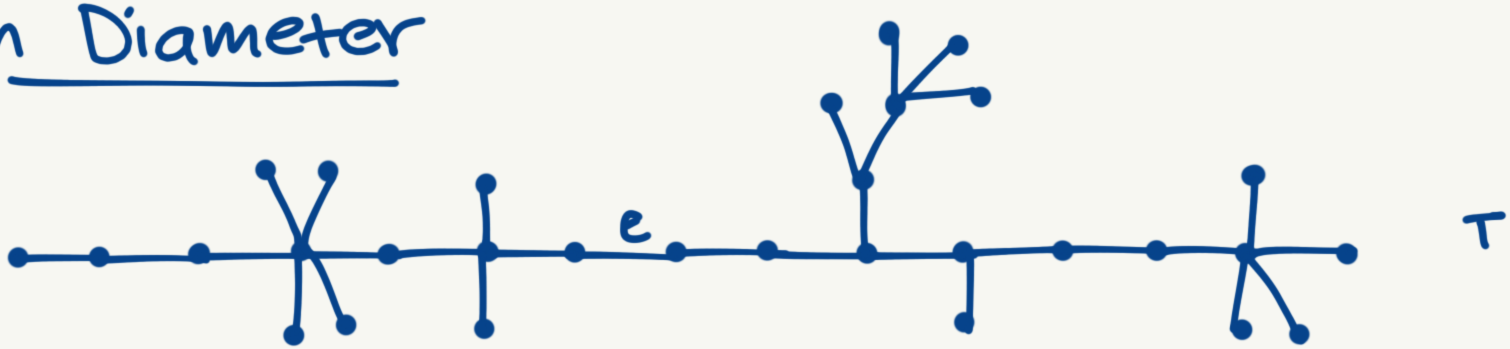

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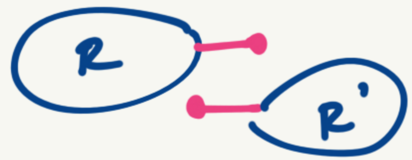
We wish to find a good pair 

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Let $\mathcal{H} = \{$ Trees R s.t nbhd around some copy of R in T has just one extra edge and vertex $\}$

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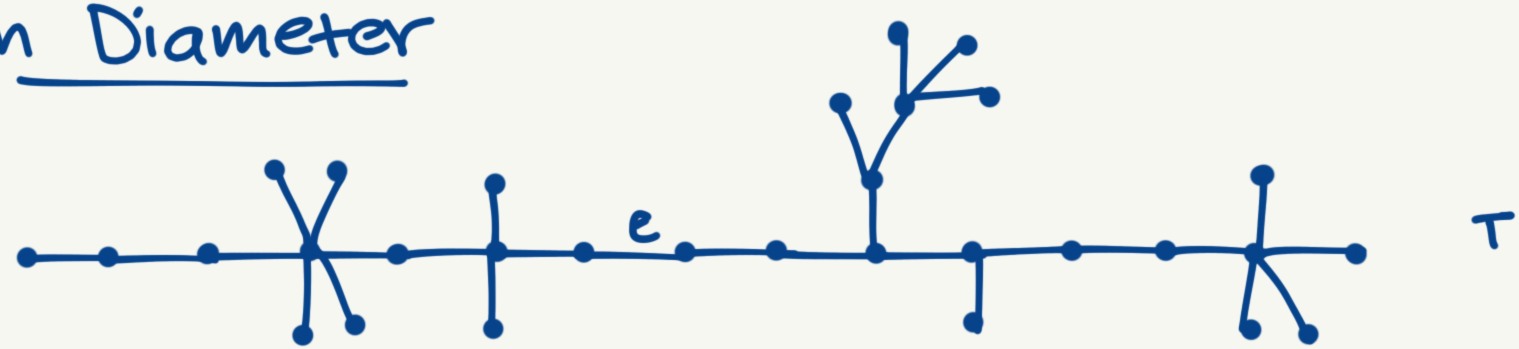

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Let $\mathcal{H} = \{$ Trees R s.t nbhd around some copy of R in T has just one extra edge and vertex $\}$

Then (R, S) with $R, S \in \mathcal{H}$ and $|V(R)| + |V(S)| = n$ are our "candidate pairs"

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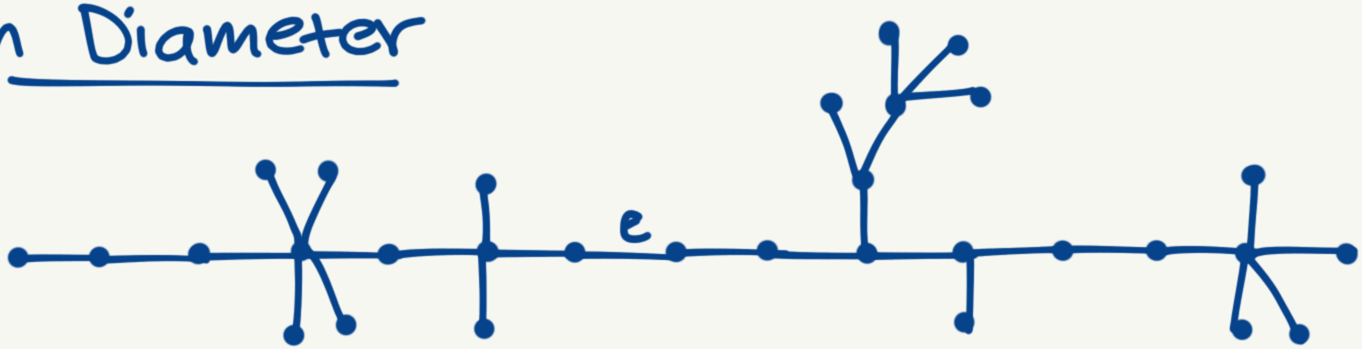

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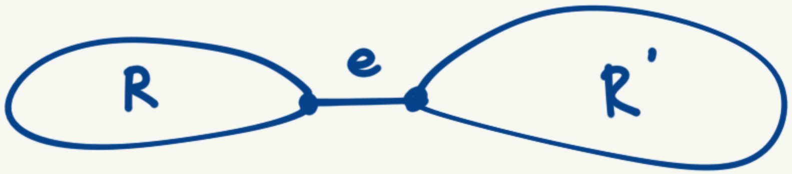

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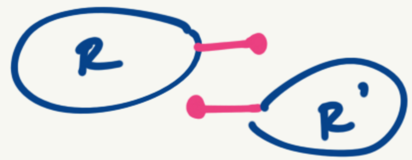

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$$\# \, R\!-\!\bullet \text{ in } T = \begin{cases} \# \, R\!-\!\bullet \text{ in } S \\ \text{OR} \\ \# \, R\!-\!\bullet \text{ in } S + \underline{1} \rightarrow \bullet\!-\!S \cong \bullet\!-\!R' \end{cases}$$

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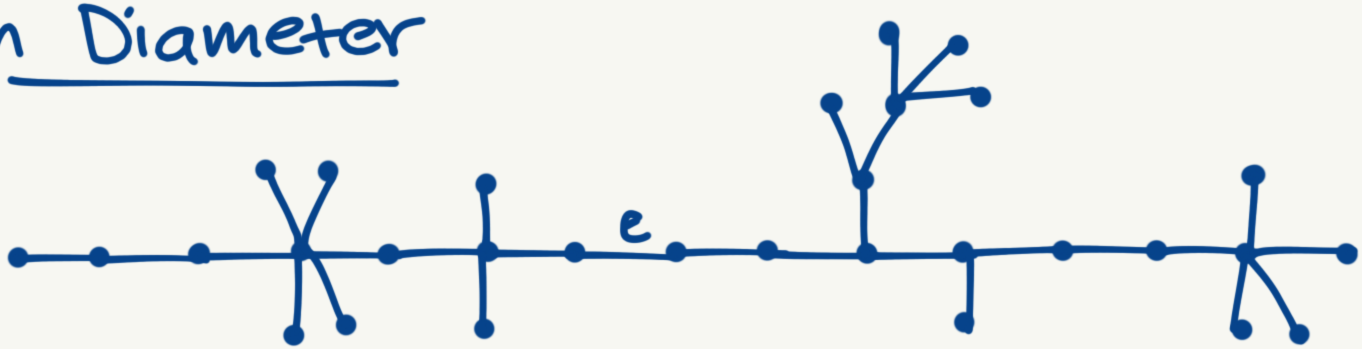

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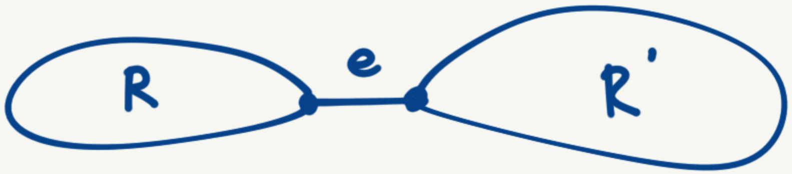
Abstractly, 

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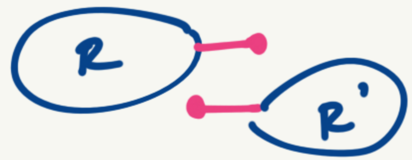
We wish to find a good pair 

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✓ Extension count

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\#  in T =

✓ Extension counting

$\Bigg\{$ \#  in S

OR

\#  in S + 1 → 

✓ Extension counting

Low Diameter

$\to$ longest path k vertices, say k odd.

Low Diameter

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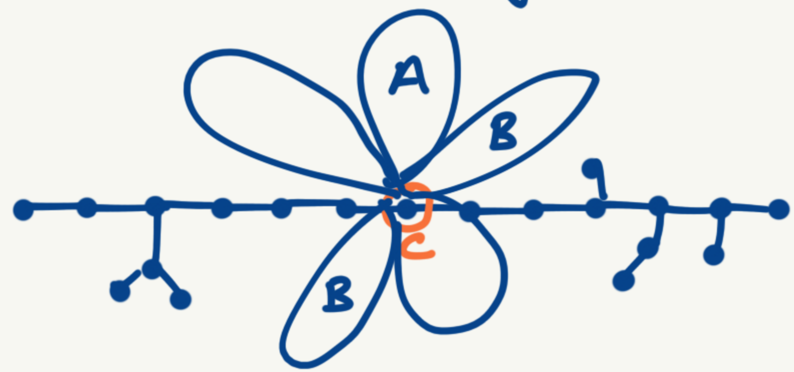
We can see P_k on a single card!



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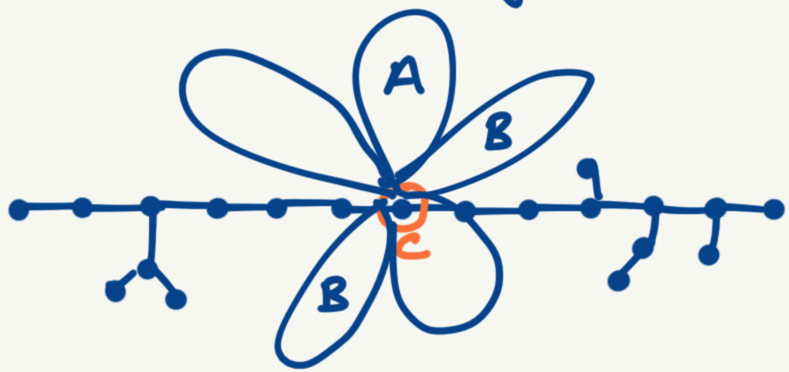

Let b be a rooted tree (candidate branch at c)

We count:

$$\begin{array}{l}\#\text{ Branches at } c \text{ iso} \\ \text{to } b\text{, counted once} \\ \text{for each longest path}\end{array} = \left(\begin{array}{c}\#\text{ Branches at } c \\ \text{iso to } b\end{array}\right) \cdot \left(\begin{array}{c}\#\text{ copies of} \\ P_k \text{ in } G\end{array}\right)$$

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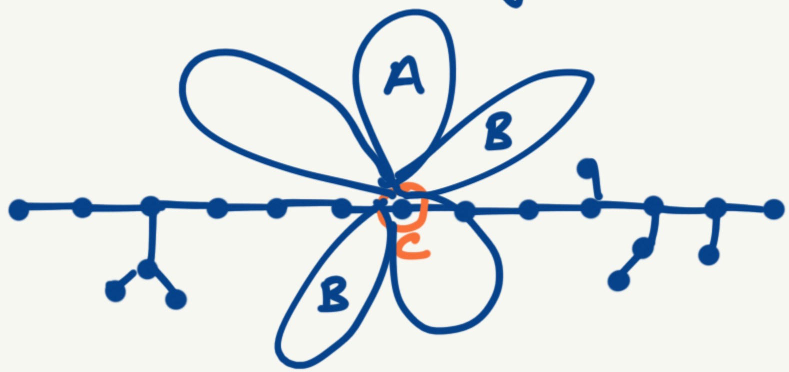

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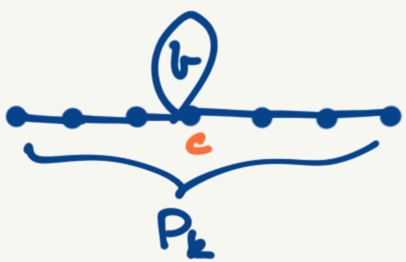
$\|$

\# 

in G + (

\# $P_{\frac{k-1}{2}-1}$ in T rooted) · (\# 

in G)

$P_{\frac{k-1}{2}+1}$

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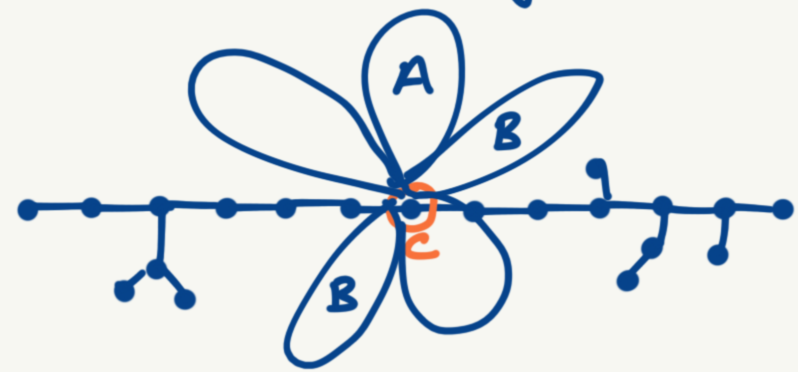

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✓ Kelly

$=$

$\#$  in G $+$

✓ G−H maximal

$\Big($  $\#\, P_{\frac{k-1}{2}-1}$ in b rooted $\Big) \cdot \Big(\#$  c in $G\Big)$

✓ Direct

✓ G−H