Reconstructing CAT(0) Cube Complexes from Boundary Distances

Jane Tan (Oxford)

with Haslegrave, Scott and Tamitegama

G G S E

What is combinatorial reconstruction?

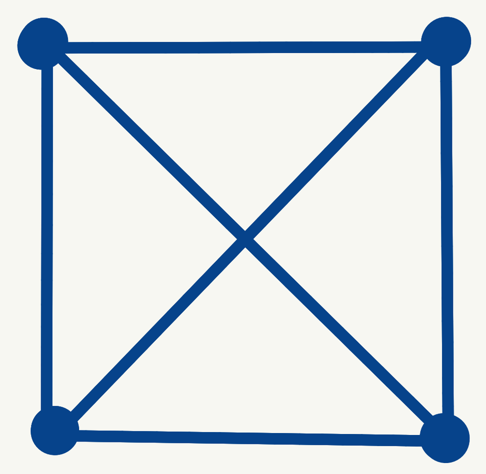
What is combinatorial reconstruction?

Given some partial (structural) information about a (discrete) mathematical object (with particular properties / within a certain family), we want to (give an algorithm to) reconstruct the object in full (up to some notion of $\sim$).

Reconstruction in graph theory

Reconstruction in graph theory

① "Classical" graph reconstruction

Deck of 

Reconstruction in graph theory

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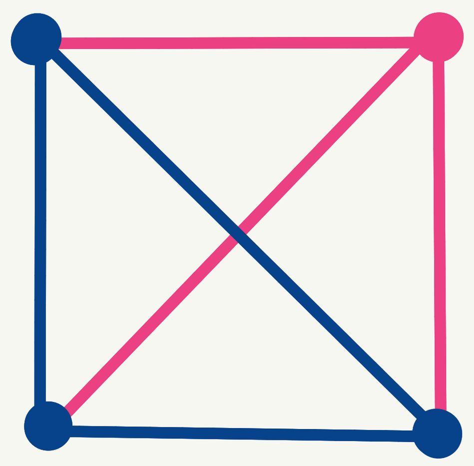
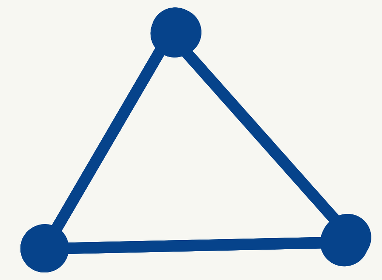
Deck of  is the multiset $\{$  ,

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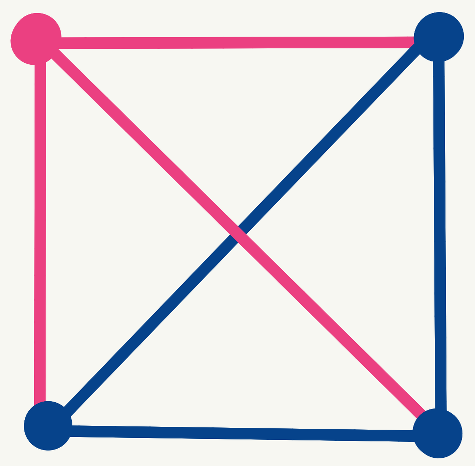
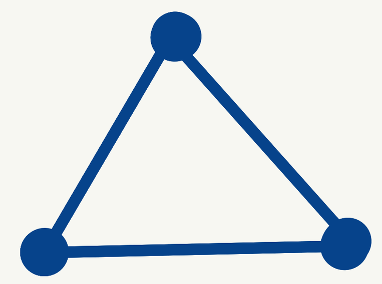
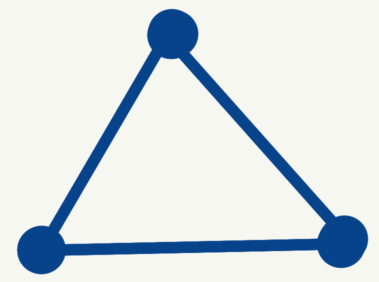
Deck of  is the multiset $\{$ , ,

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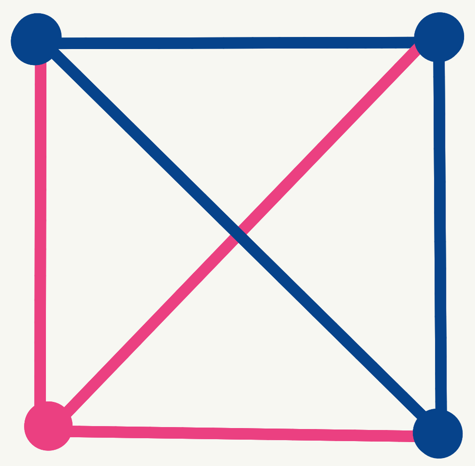
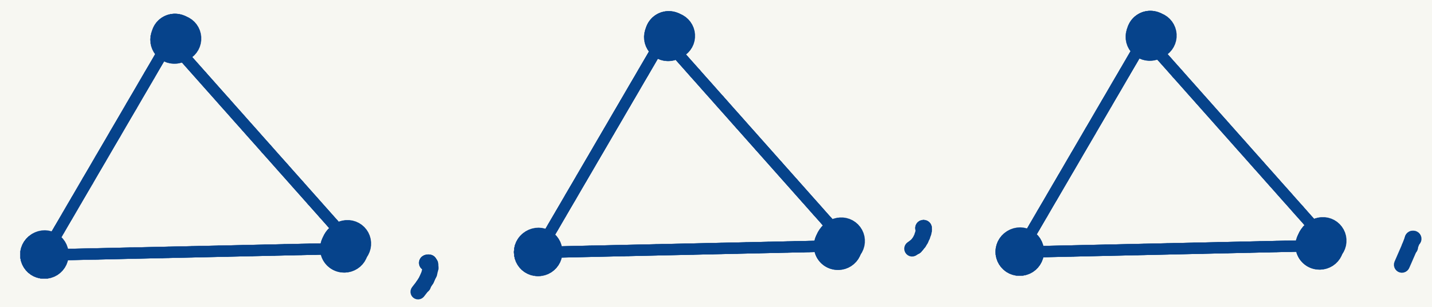
Deck of  is the multiset $\{$  ,

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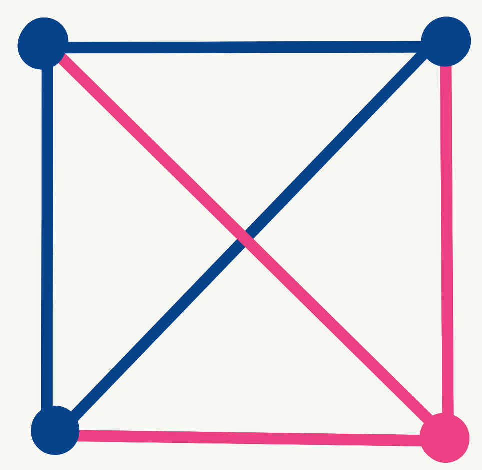
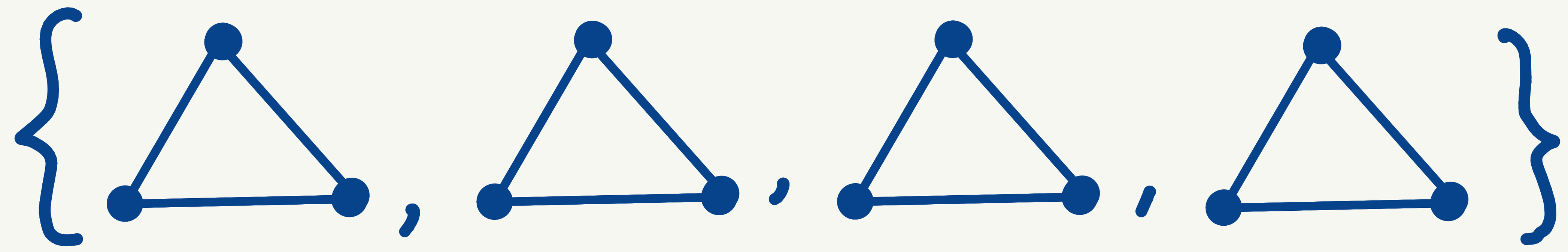
Deck of  is the multiset 

Reconstruction in graph theory

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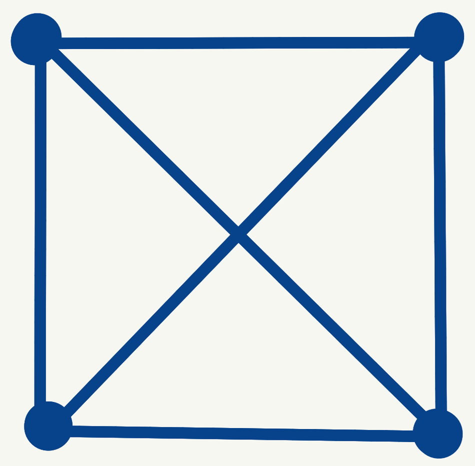
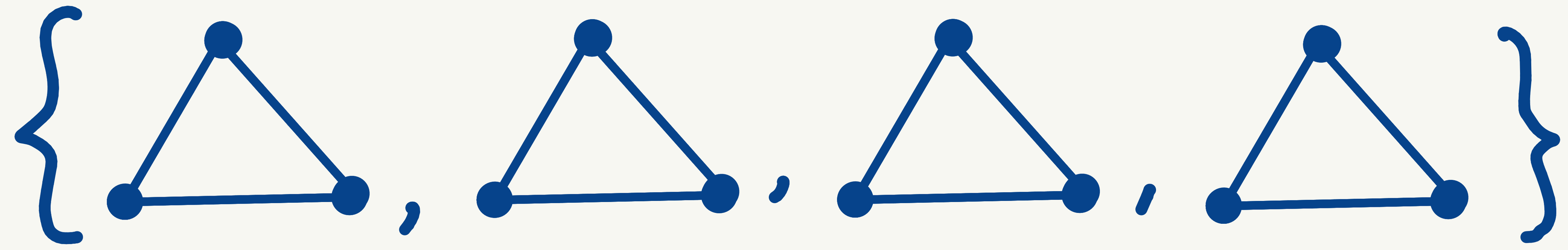
Deck of  is the multiset 

Conjecture (Kelly, Ulam 1942):

Every graph with ≥ 3 vertices is uniquely (up to isomorphism) determined by its deck.

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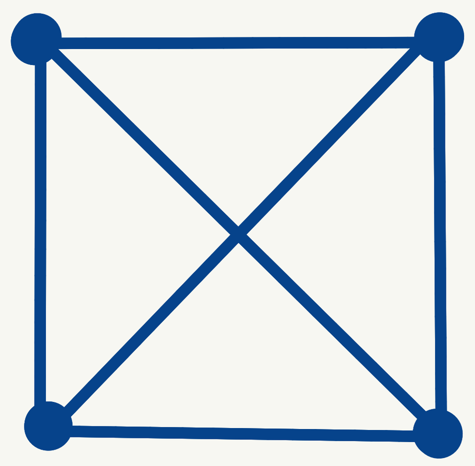
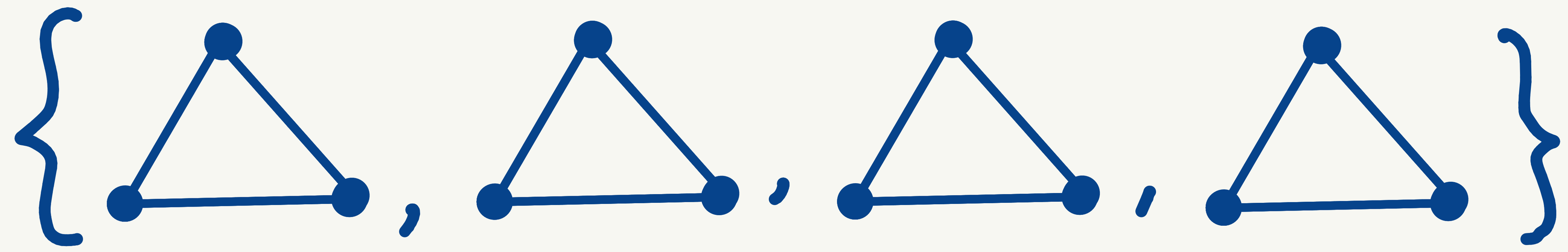
Deck of  is the multiset 

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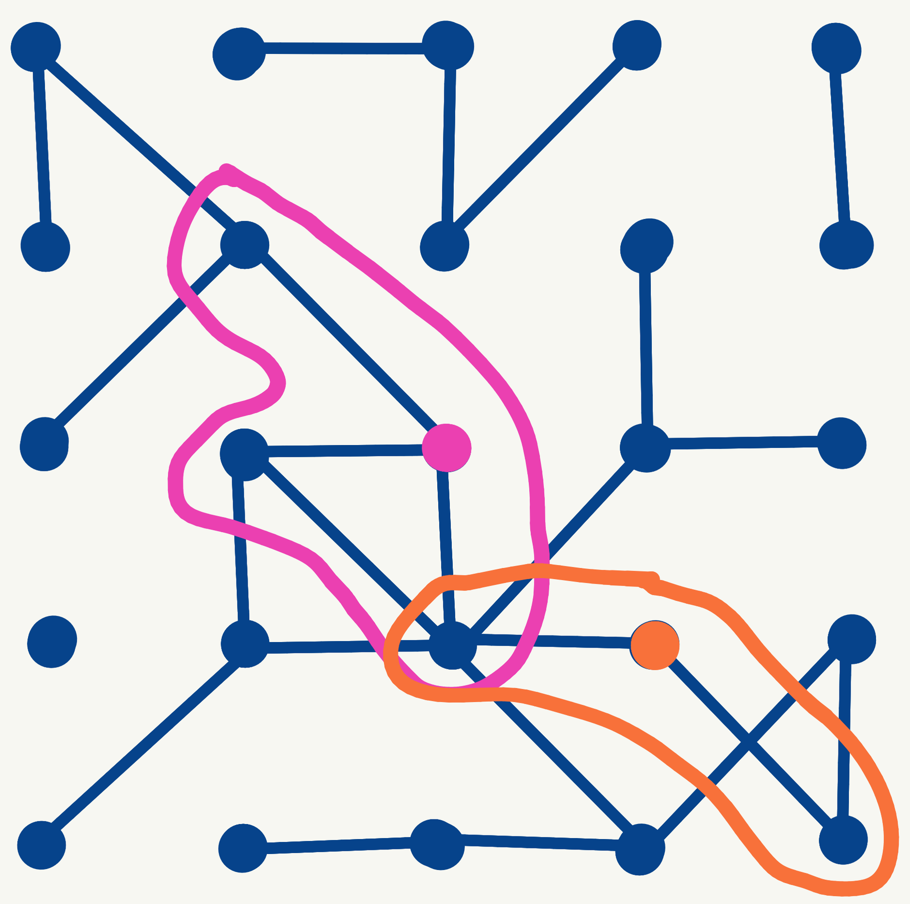
② Shotgun assembly (Mossel, Ross 2022)



1-neighbourhoods

Which graphs are reconstructible from their collection of r-neighbourhoods?

(Positive results for random graphs)

Reconstruction in graph theory

③ Trees via leaf distances

Thm (Zarečkiĭ 1965, Buneman 1974)

Rooted binary trees can be reconstructed from pairwise distances between leaves.

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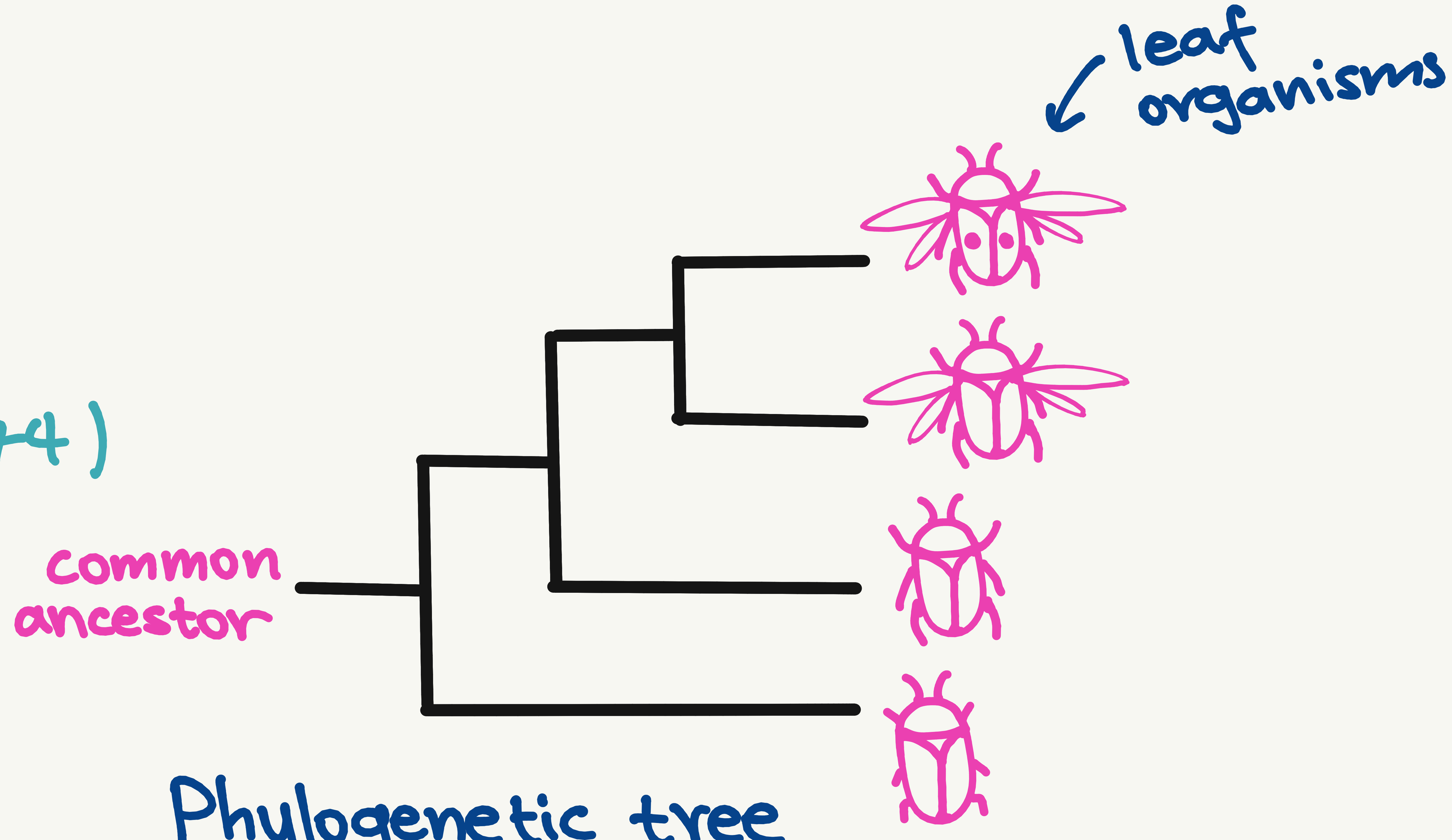

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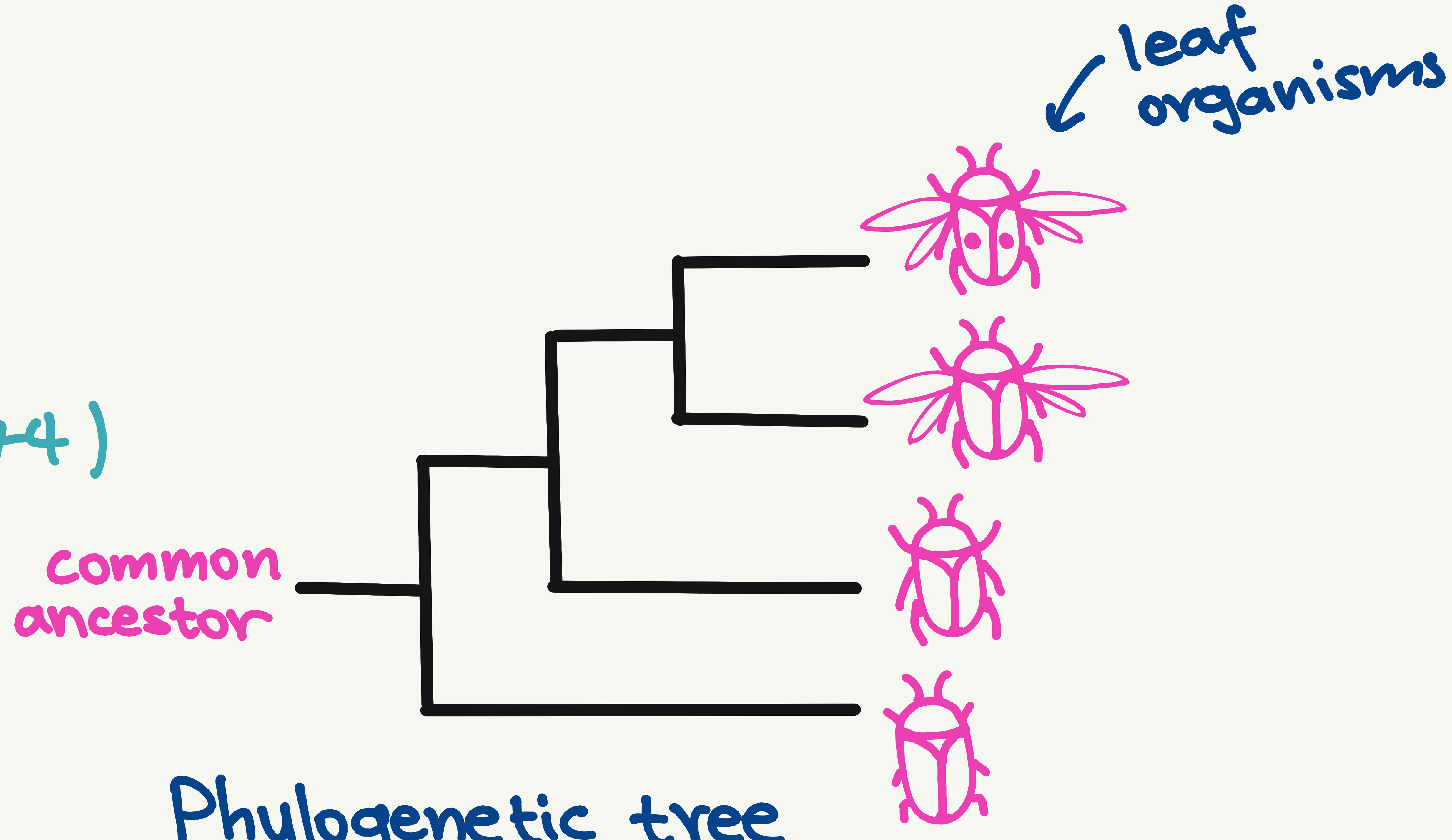

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④ Convex polytopes

conjectured by Perles

Thm (Blind, Mani 1987 / Kalai 1988)

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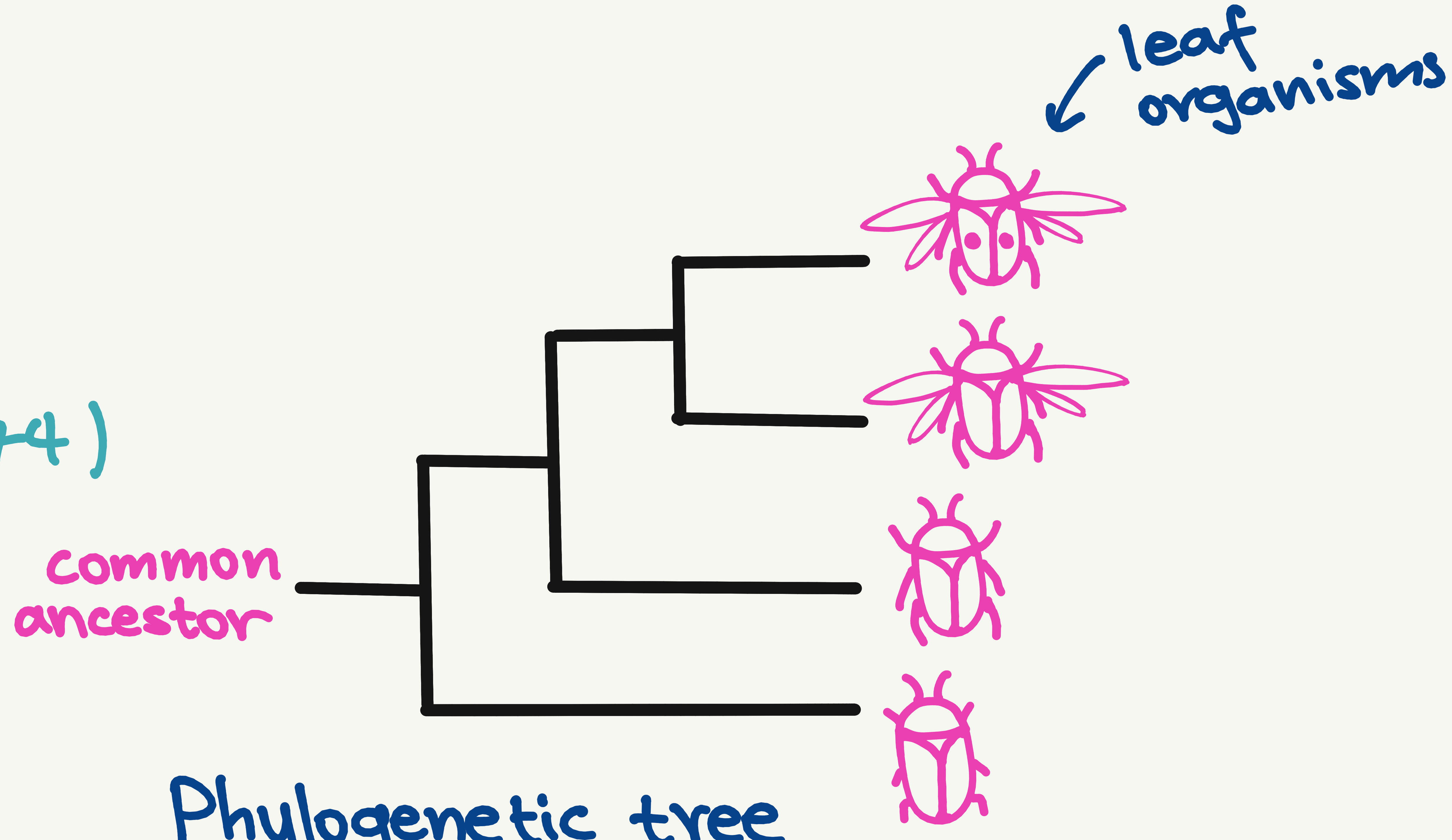

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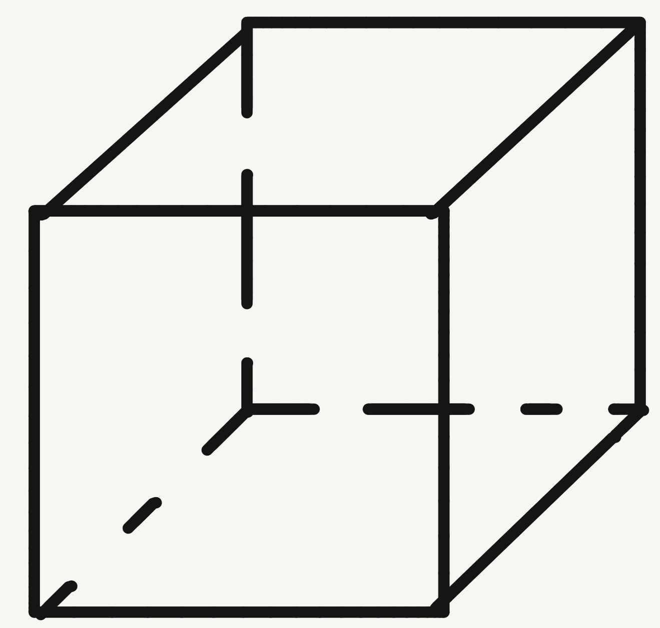
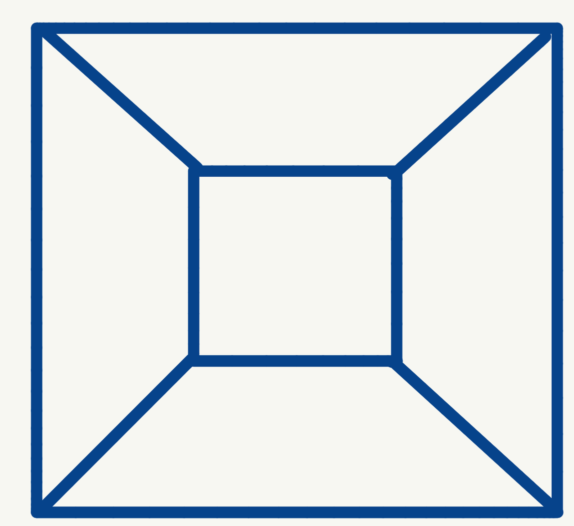
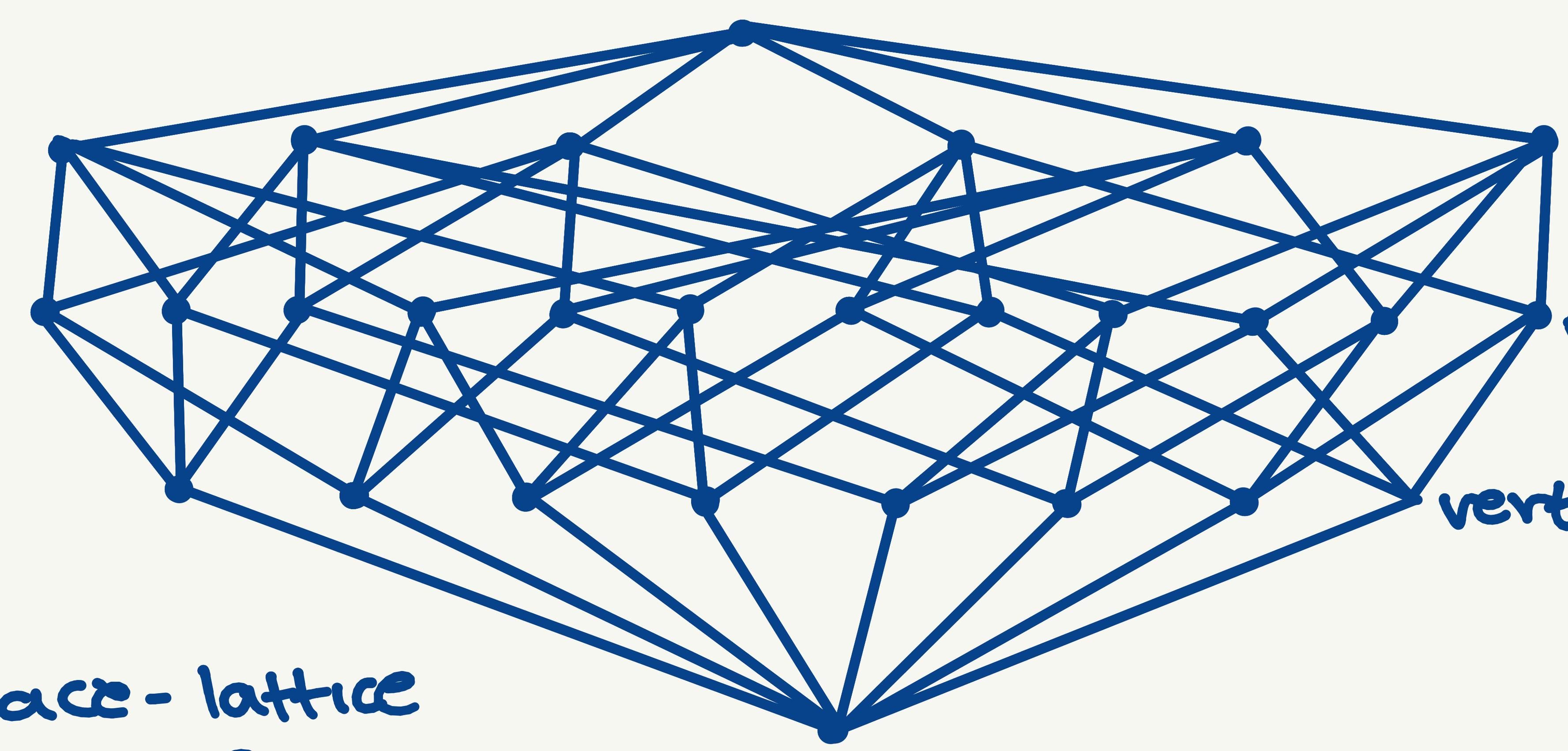


P



Graph of P



Face-lattice of P

From differential geometry.

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Def$^{\underline{n}}$ A Riemannian manifold (M, g) is boundary rigid if its associated metric $d_g : M \times M \to \mathbb{R}$ is determined up to isometry by its boundary distance function $d_g|_{\partial M \times \partial M}$.

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Generally wide open in dimensions ≥ 3

A discrete version?

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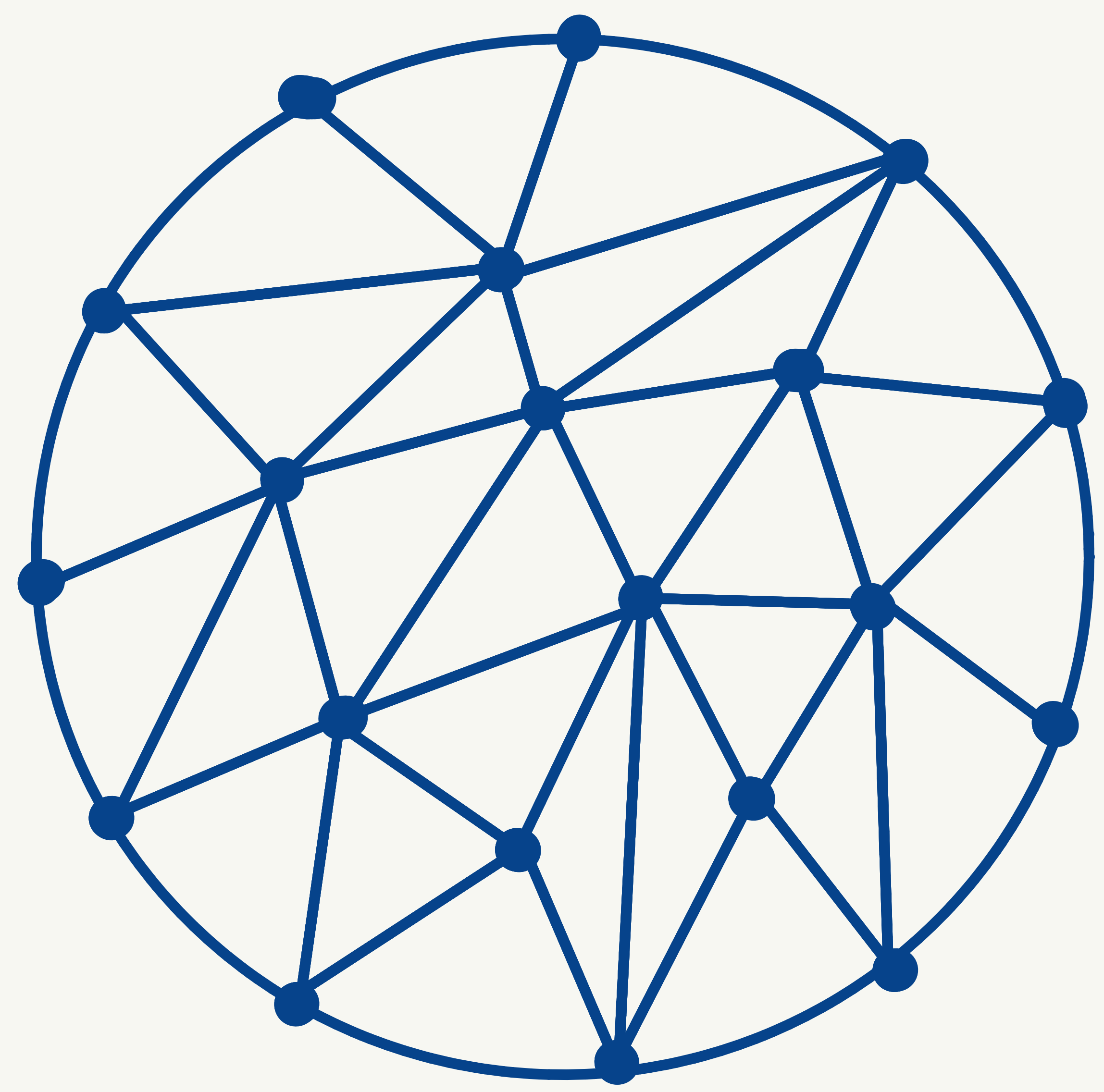
Conjecture (Benjamini)

For every near-triangulation with all internal degrees ≥ 6, one can reconstruct the graph from the matrix of pairwise distances between all vertices on the boundary.



A near-triangulation

A discrete version

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Theorem (Haslegrave)

- The above conjecture is true

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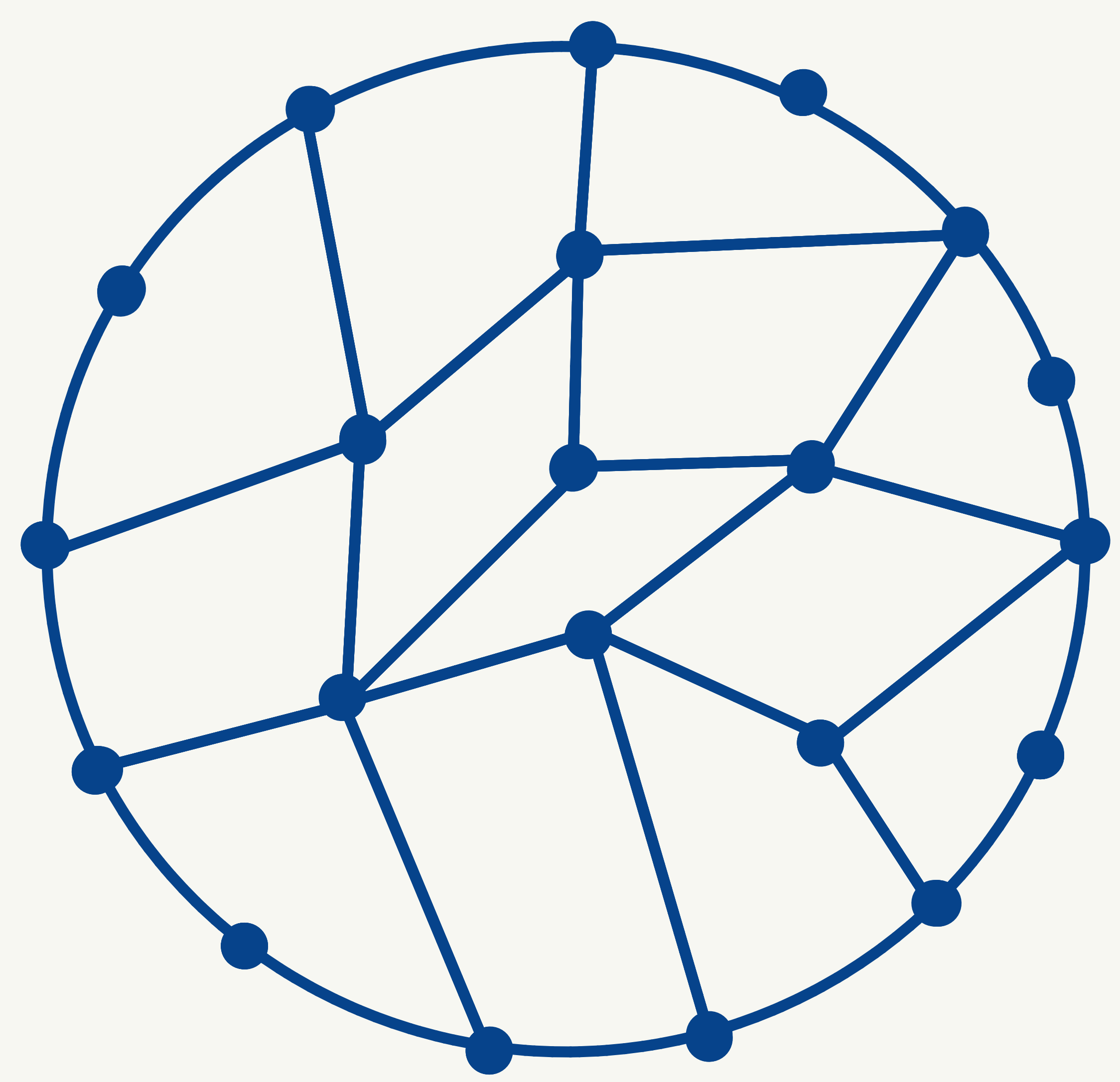
Theorem (Haslegrave)

- The above conjecture is true
- An analogous result is true for quadrangulations

A discrete version

Theorem (Haslegrave 2021)

Suppose that Q is a quadrangulation with simple closed boundary such that all internal vertices have degree ≥ 4. Then Q is reconstructible up to graph isomorphism from the distances between boundary vertices of Q.



A quadrangulation
with simple closed boundary

A discrete version

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$$\begin{array}{cc}
 & \begin{matrix} a & b & c & d & e & f & g & h \end{matrix} \\
\begin{matrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{matrix} &
\begin{bmatrix}
0 & 1 & 2 & 3 & 2 & 3 & 2 & 1 \\
1 & 0 & 1 & 2 & 3 & 4 & 3 & 2 \\
2 & 1 & 0 & 1 & 2 & 3 & 2 & 3 \\
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2 & 3 & 2 & 1 & 0 & 1 & 2 & 3 \\
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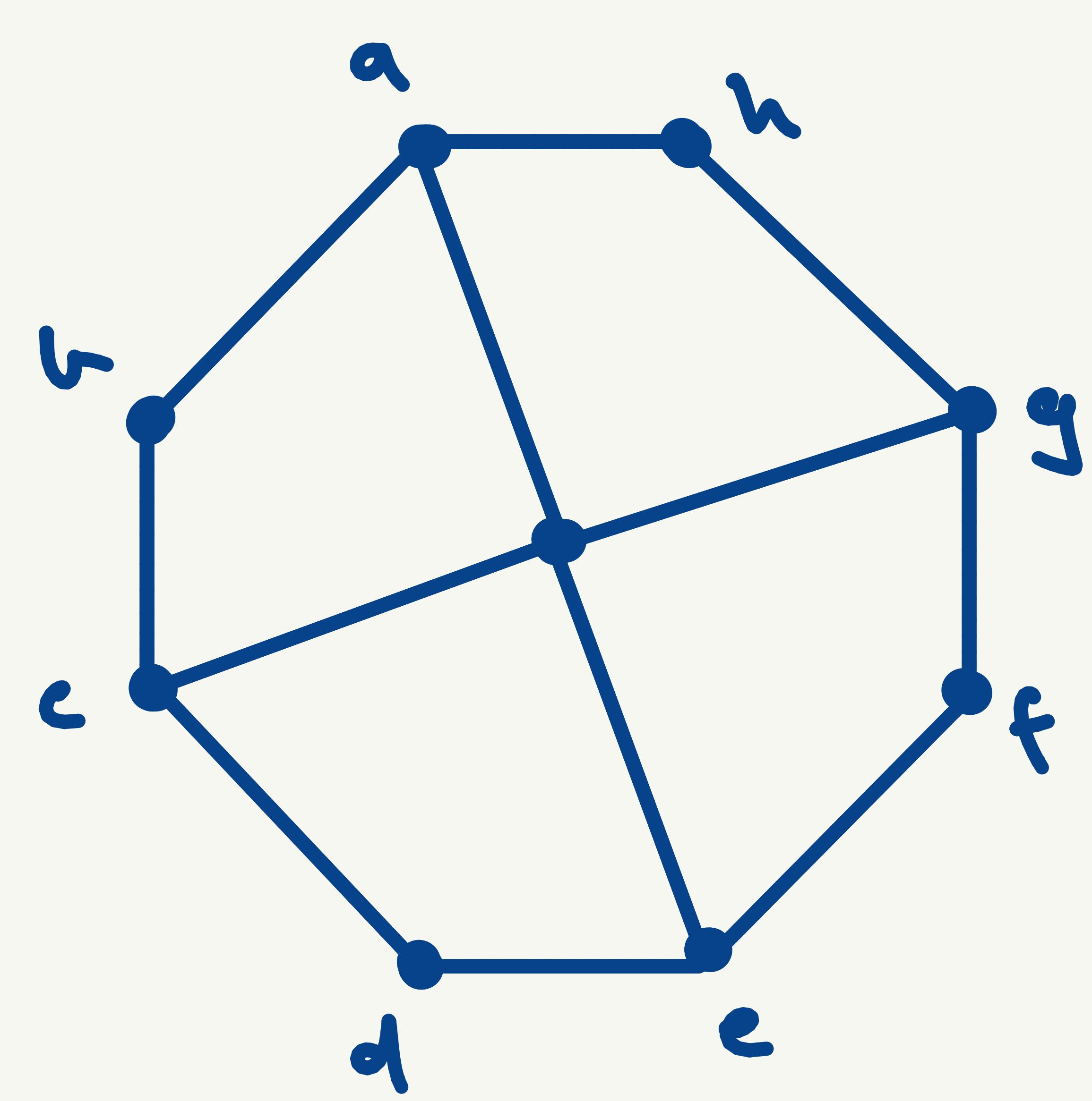

(vertices not necessarily given in cyclic order)

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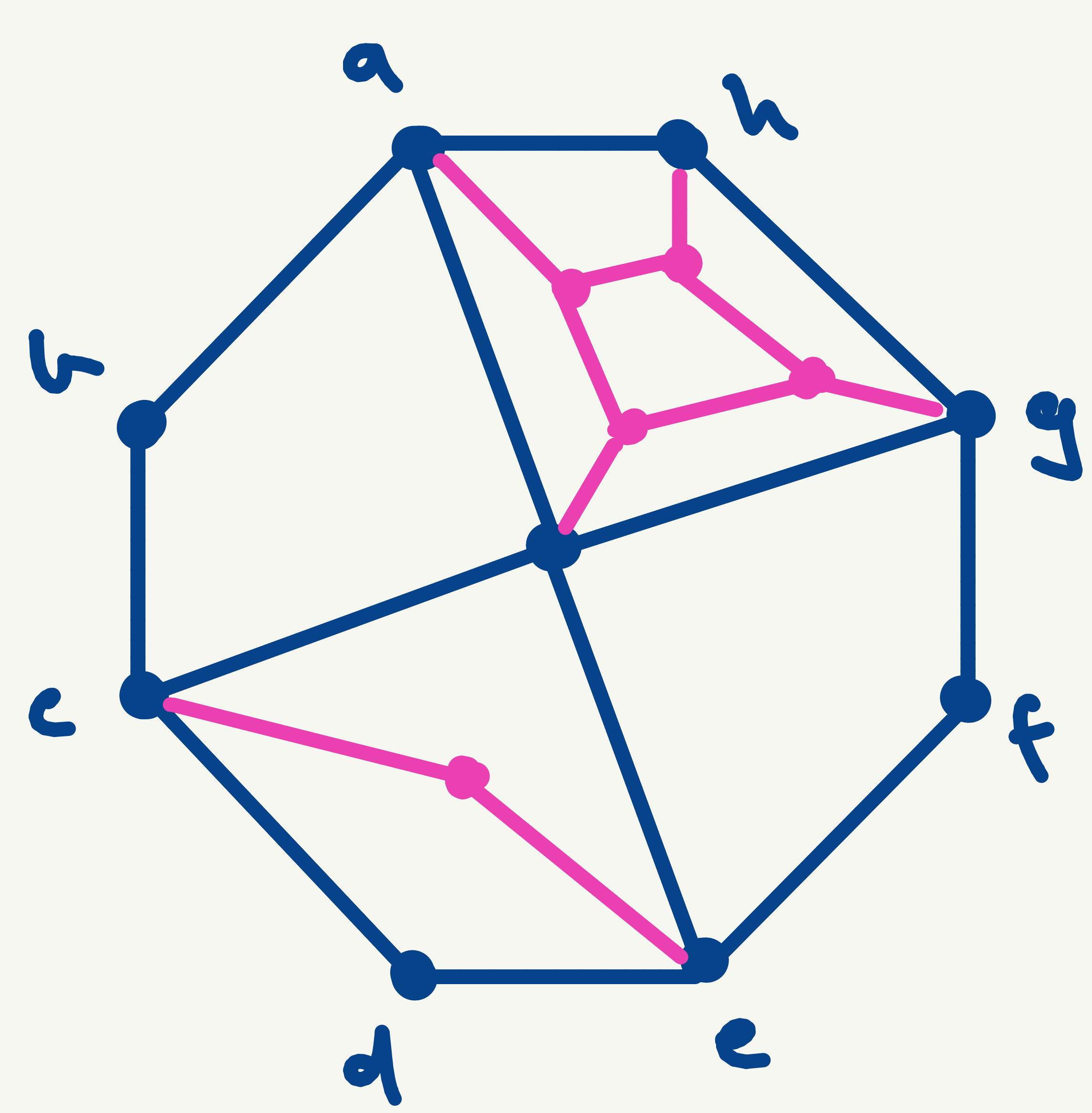

Distances between boundary vertices remain the same

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Goal: move up a dimension

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↳ what objects and conditions are we working with?

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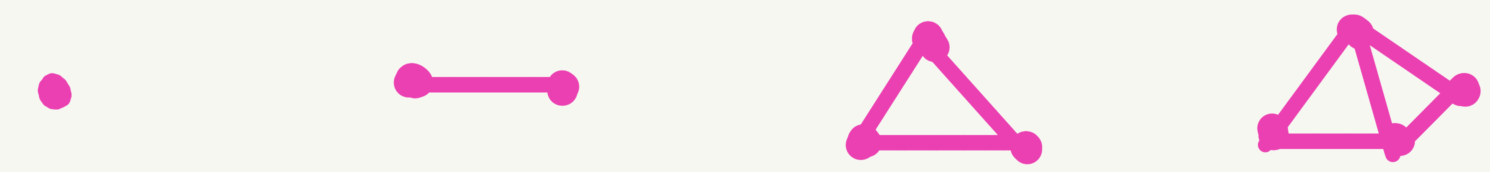
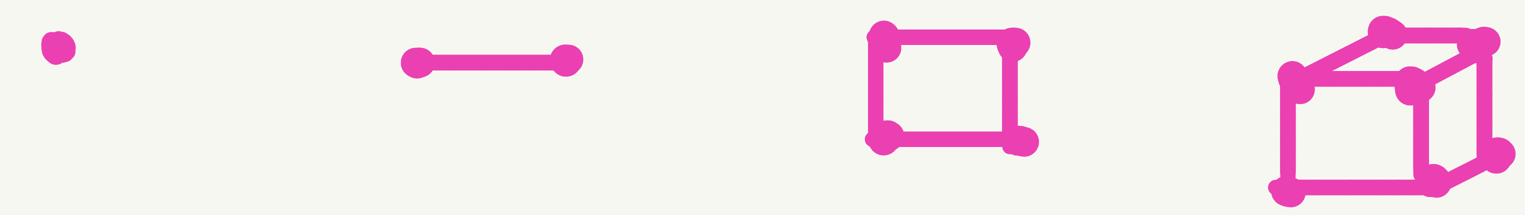
↳ what objects and conditions are we working with?

Qudrangulation ↔ cube complex

Graph isomorphism ↔ combinatorial equivalence

Degree condition ↔ curvature condition [CAT(0)]

Cube complex essentials

Dimension	0	1	2	3	n
simplex					convex hull of n+1 pts
cube					I^n

Aside

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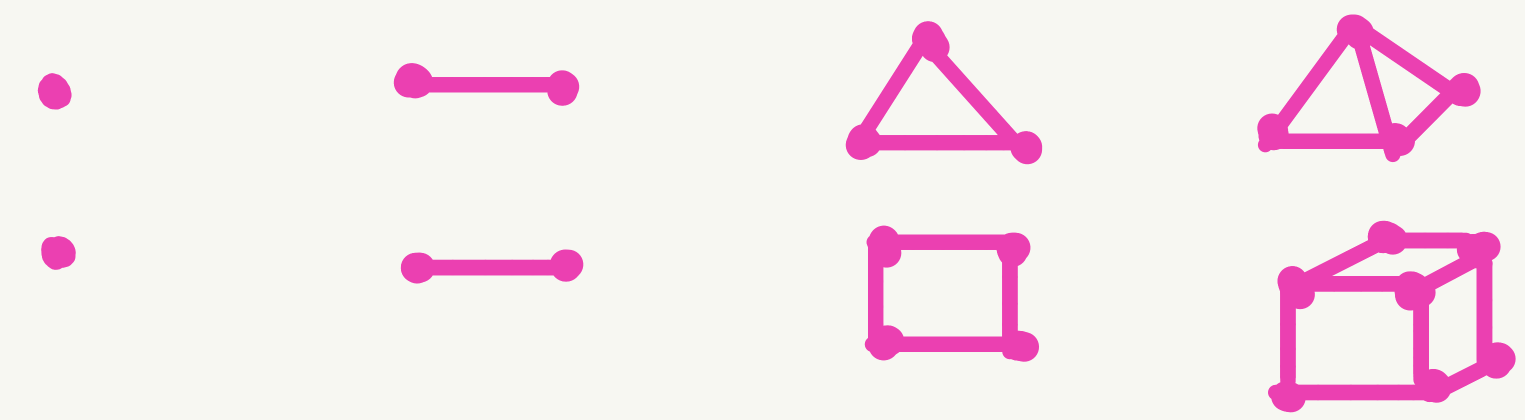

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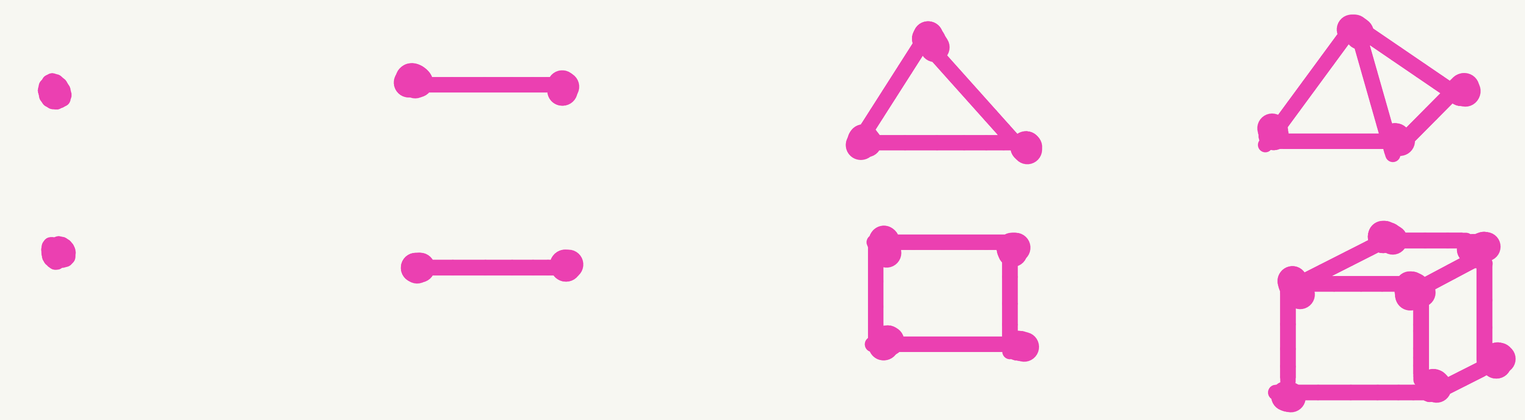

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Its **dimension** is the dimension of the top cube.

It is **pure** if every cube is in a top-dimensional cube.

The **k-skeleton** is the union of all cubes with dim $\leq k$.

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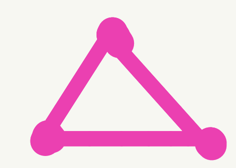
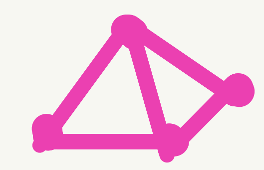
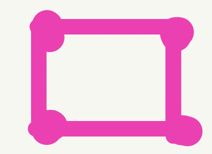
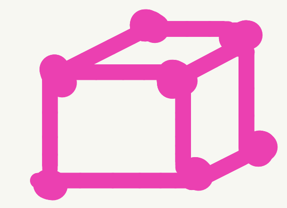
simplex	•	—			convex hull of $n+1$ pts
cube	•	—			I^n

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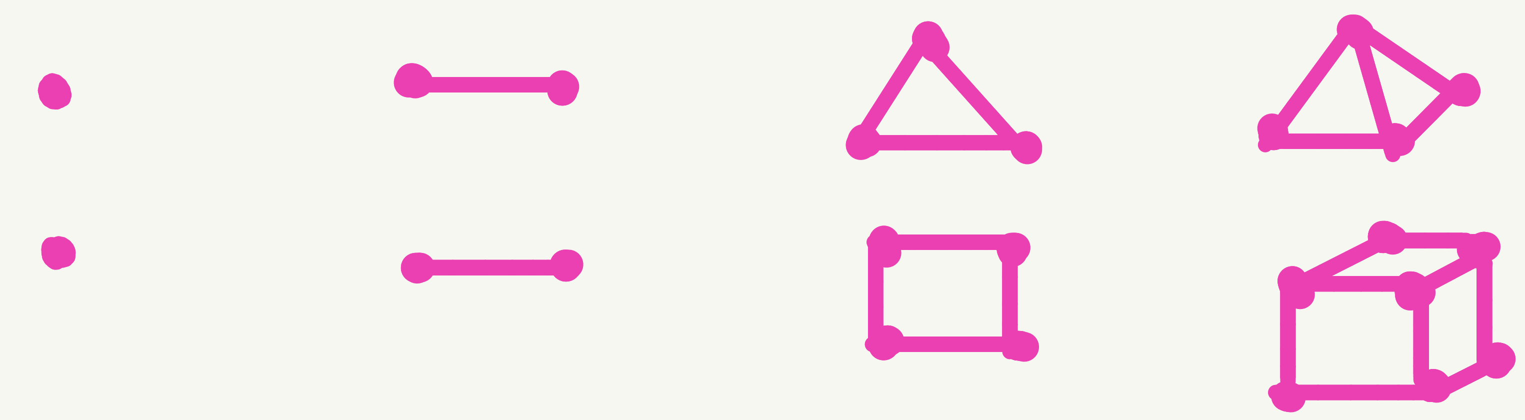

(In 3D: vertex edge face cube)

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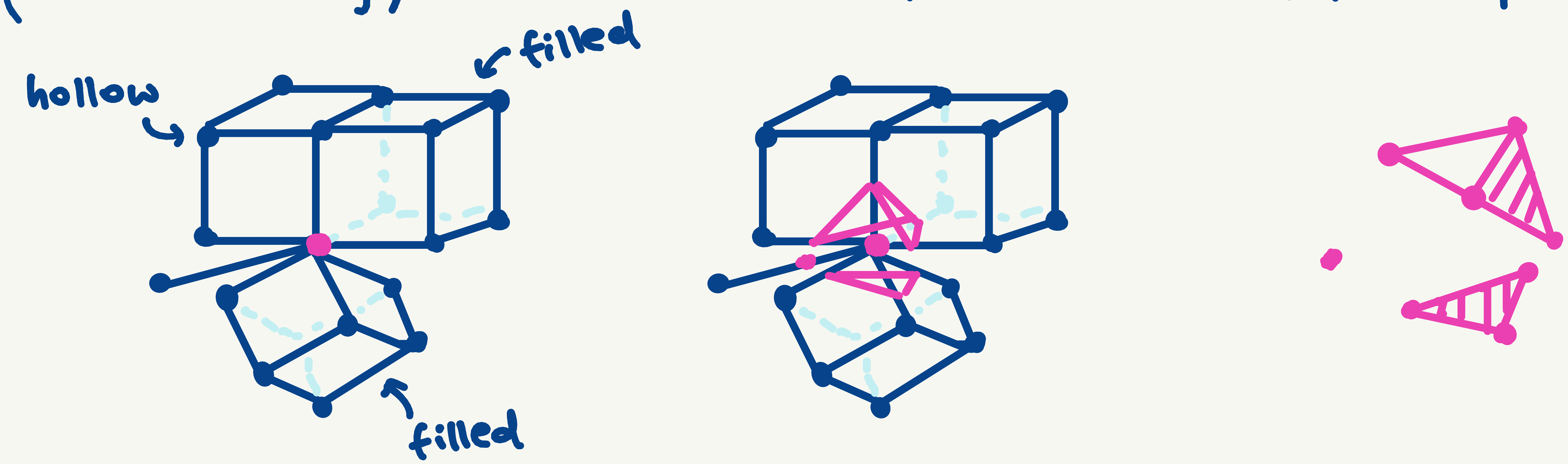
(Alternatively, intersection of $B_\varepsilon(v)$ with complex)



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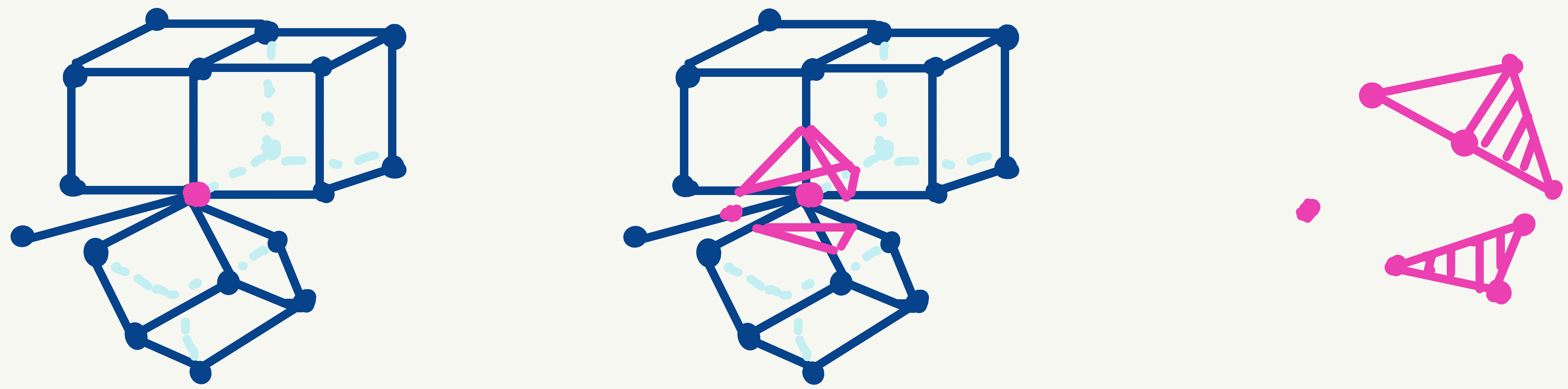

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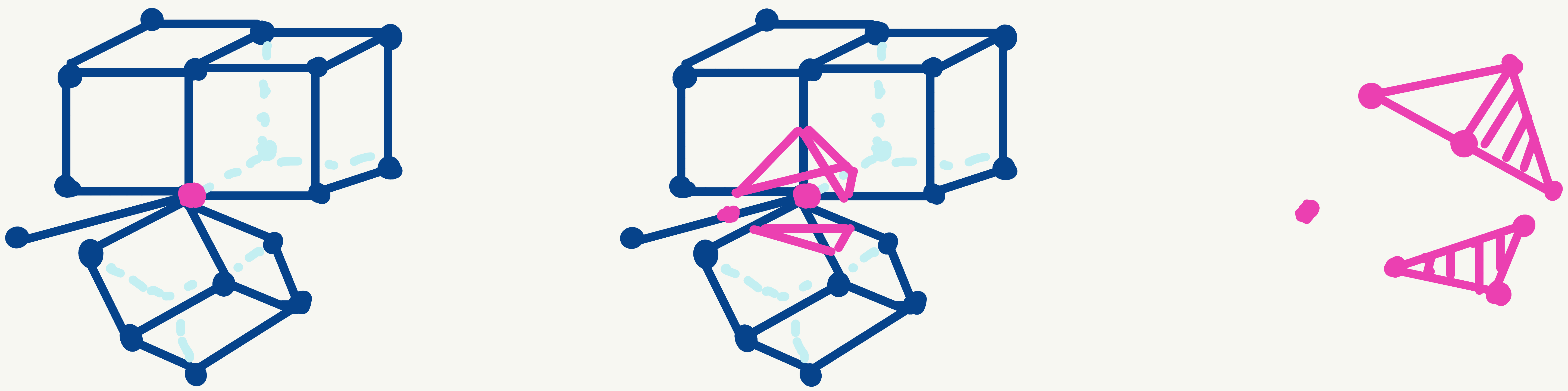

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A cube complex is CAT(0) if it is simply connected and the link of every vertex is a flag simplicial complex.

"Gromov's link condition"
1987

A discrete version

Theorem (Haslegrave, Scott, Tamitegama, T. 2023)

Suppose that X is a finite CAT(0) cube complex which admits an embedding in $\mathbb{R}^3$. Then X is reconstructible up to combinatorial type from the matrix of pairwise distances between vertices on ∂X.

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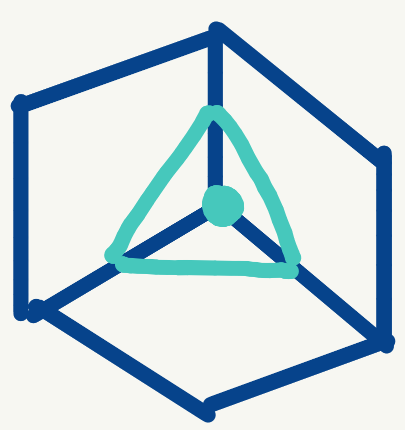
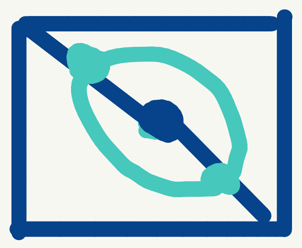
Claim: X is pure 2D, CAT(0) cube complex $\Leftrightarrow$ quadrangulation with all internal degrees ≥ 4.

Pf: pure 2D $\leftrightarrow$ quadrangulation

Gromov $\leftrightarrow$ degree condition

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in the graph metric

Boundary in the topological sense, induced by embedding in $\mathbb{R}^3$.

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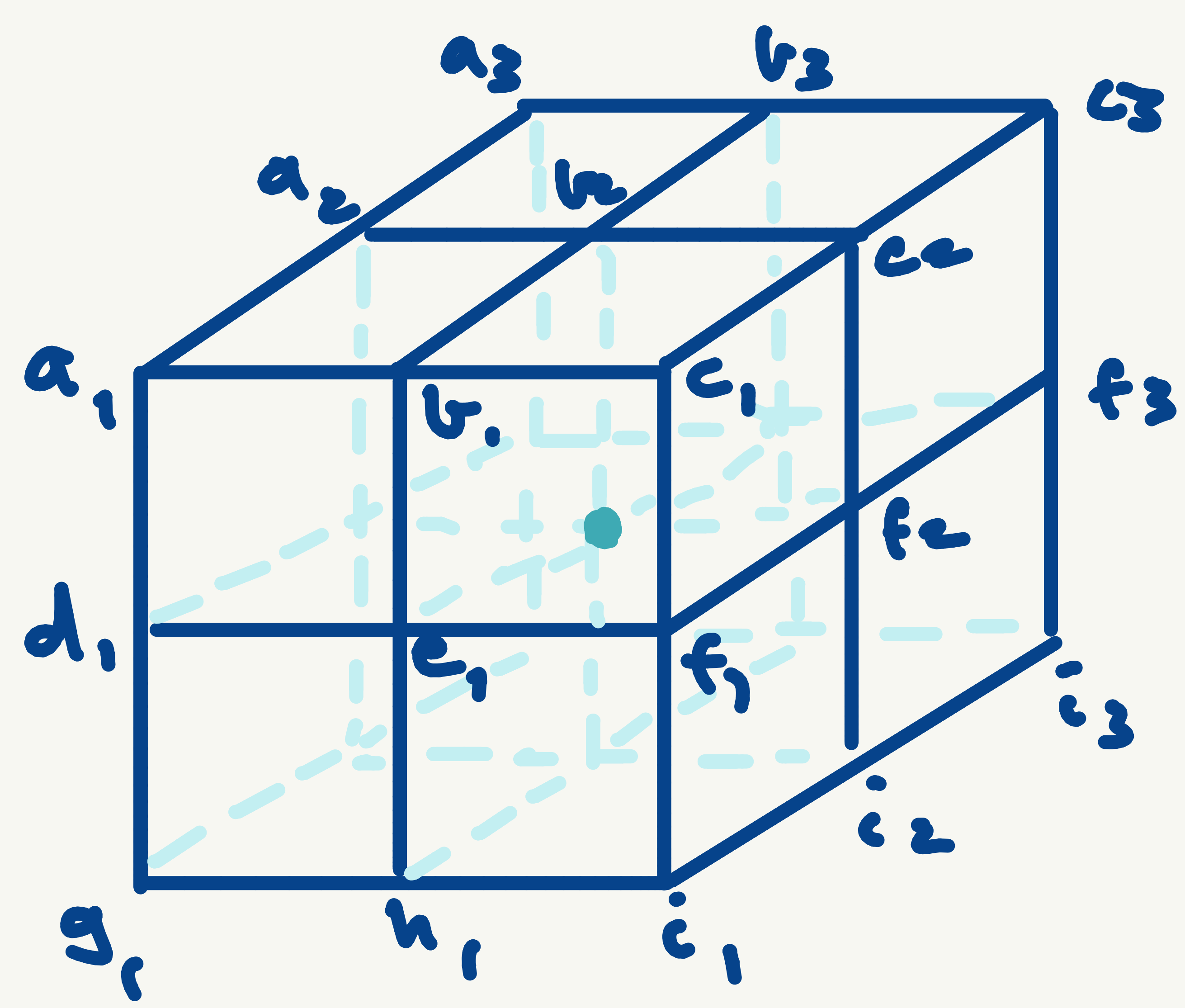
in the graph metric

Boundary in the topological sense, induced by embedding in $\mathbb{R}^3$.

$$\begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ i_2 \end{array} \begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & a_3 & \cdots & i_2 \end{array} \\ \left[\begin{array}{ccccc} & 1 & 2 & & 6 \\ & & 1 & & 5 \\ & & & & 4 \\ & & & & \\ & & & & \end{array}\right] \end{array} \longrightarrow$$

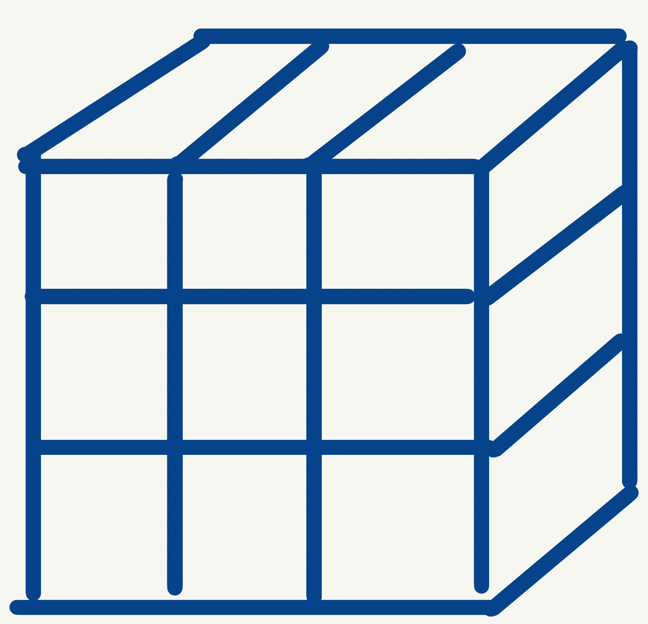


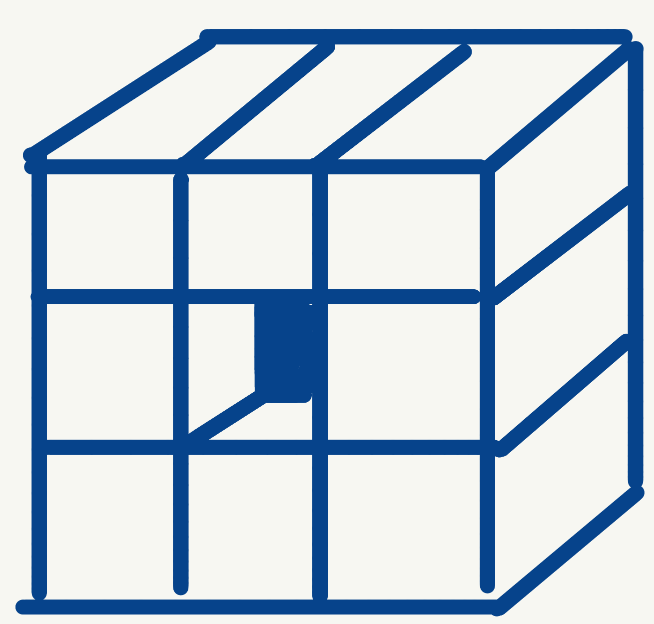
A discrete version

Theorem (Haslegrave, Scott, Tamitegama, T. 2023)
Suppose that X is a finite CAT(0) cube complex which admits an embedding in $\mathbb{R}^3$. Then X is reconstructible up to combinatorial type from the matrix of pairwise distances between vertices on ∂X.

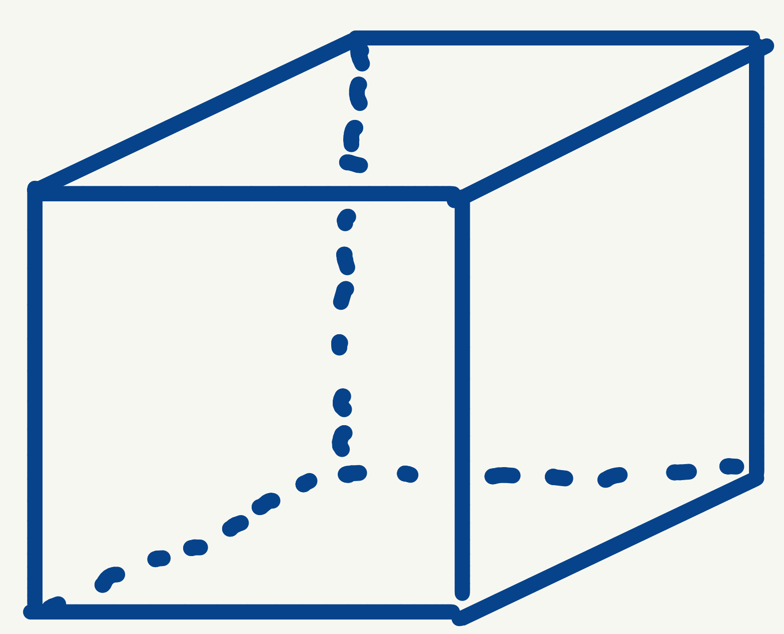
This is necessary:

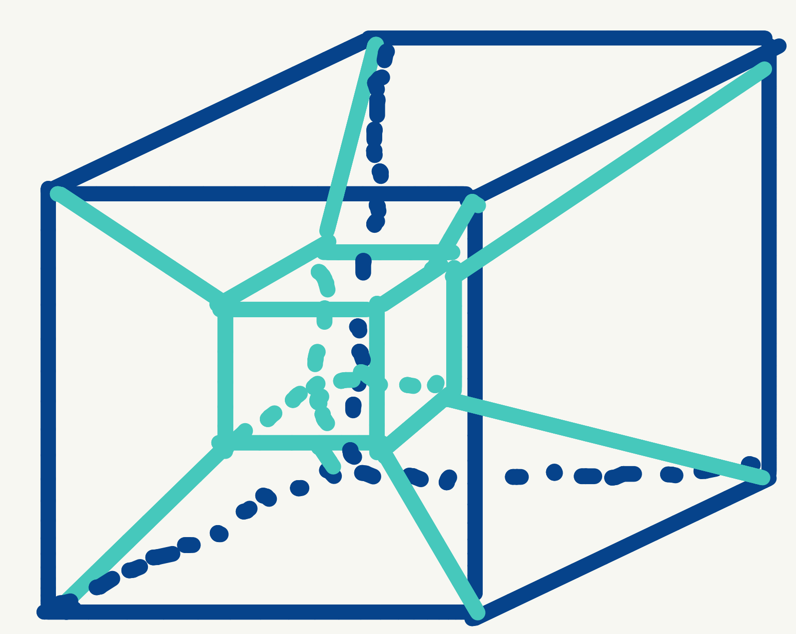
Remove simply connected





Remove flag





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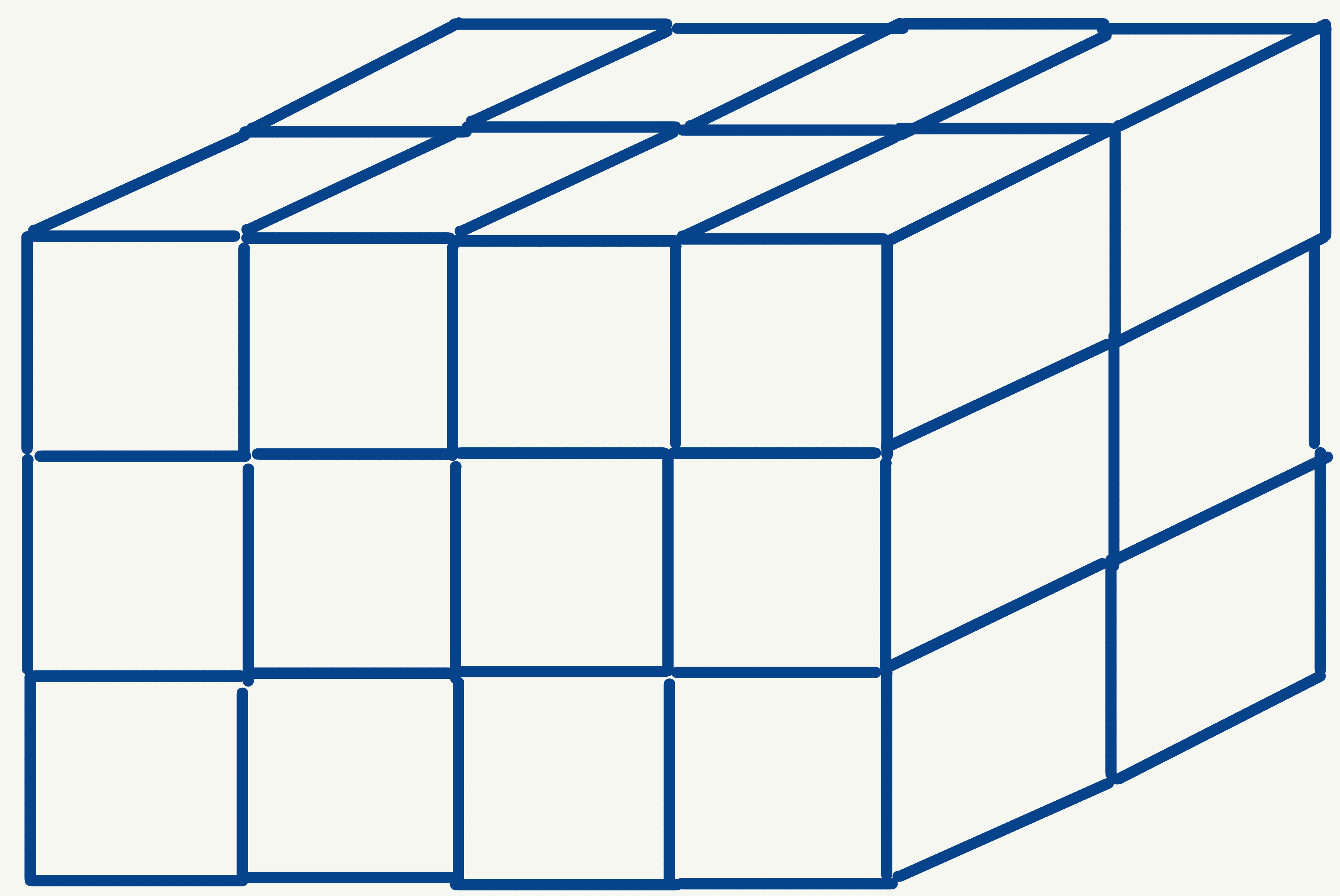

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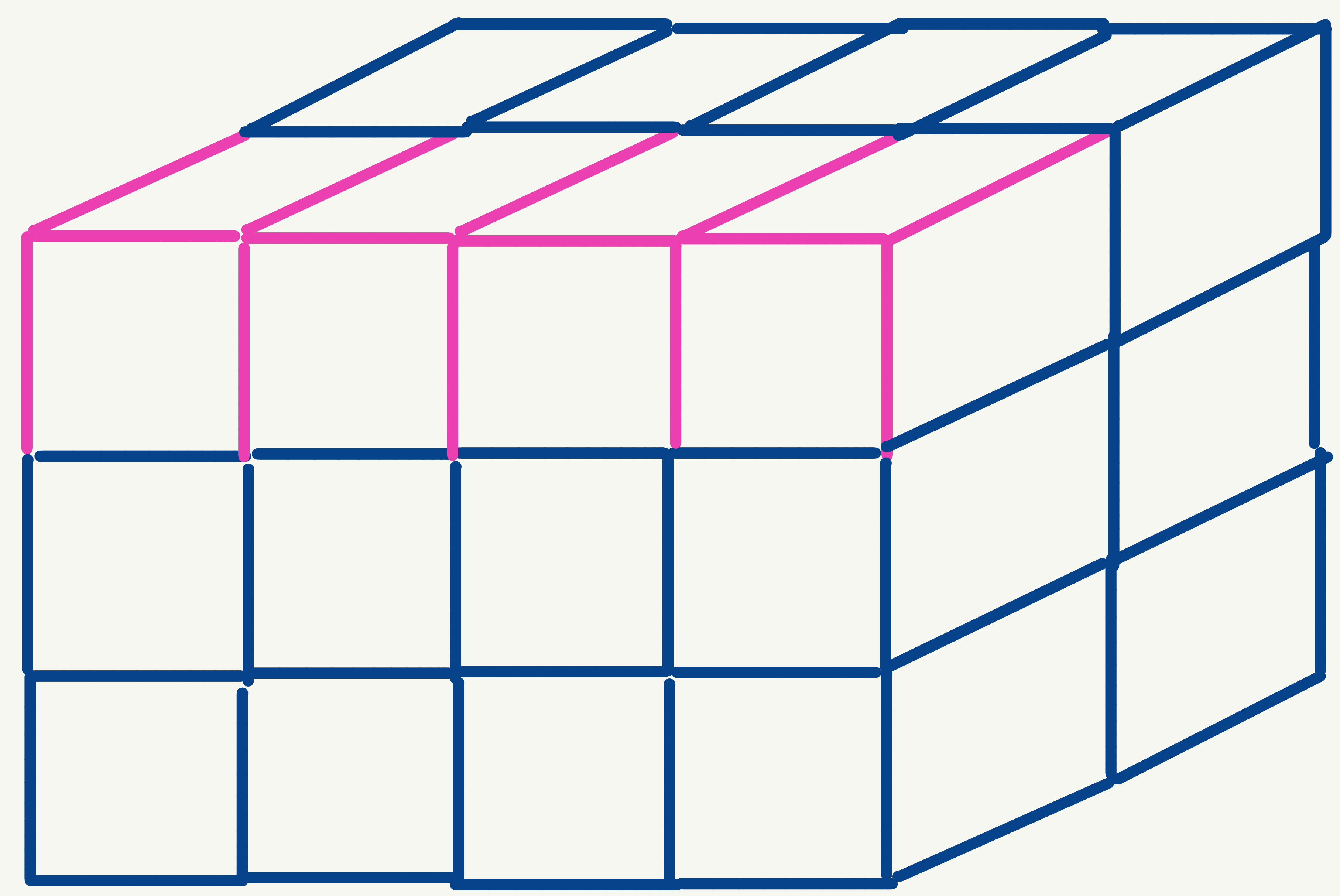

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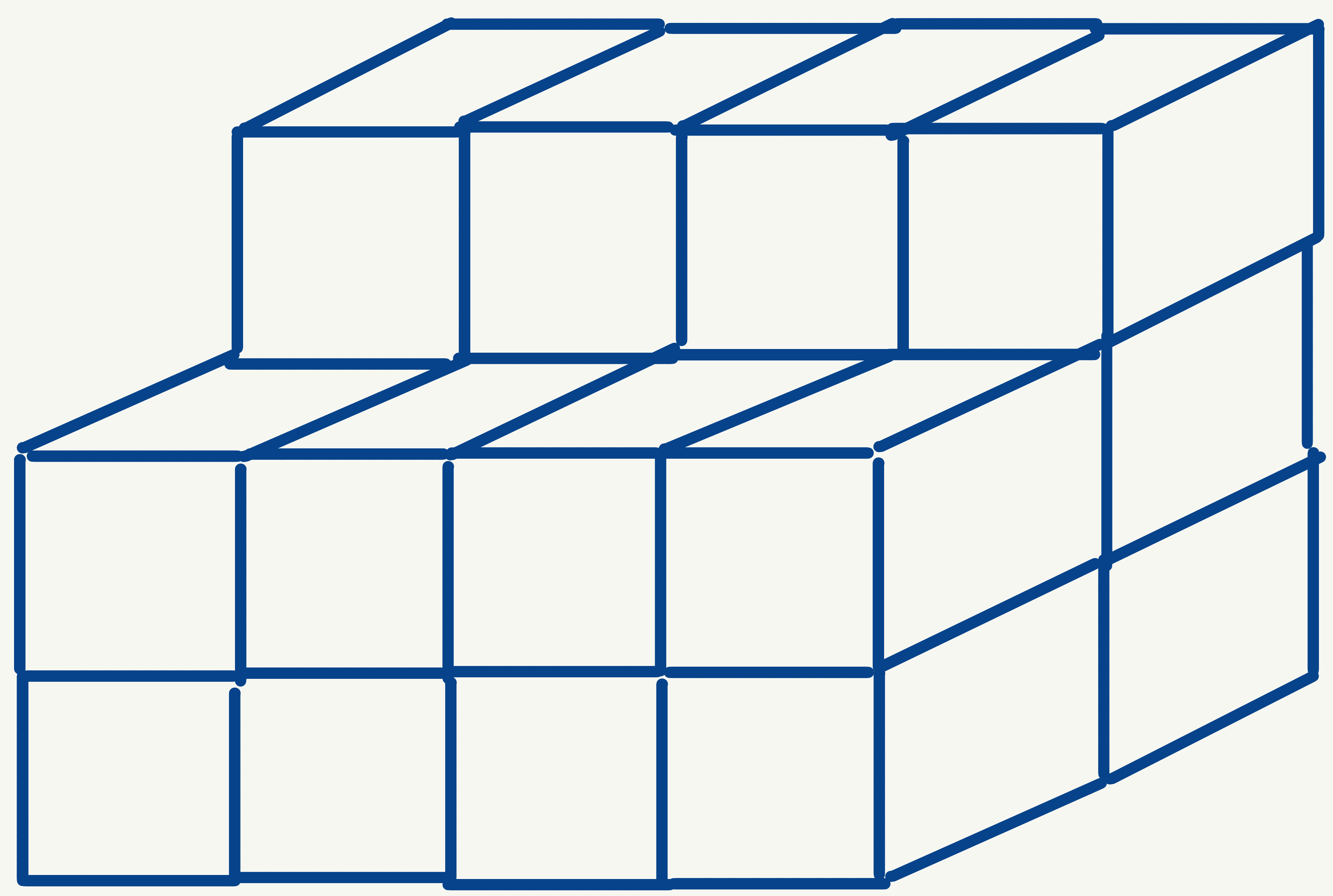

Key Lemmas & Outline

Structure lemma: Every finite CAT(0) cube complex embeddable in $\mathbb{R}^3$ that is not a single edge contains one of these 4 structures

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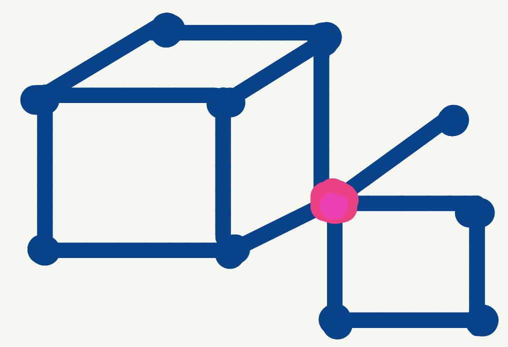

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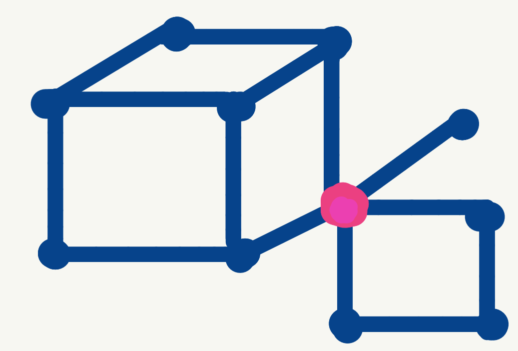
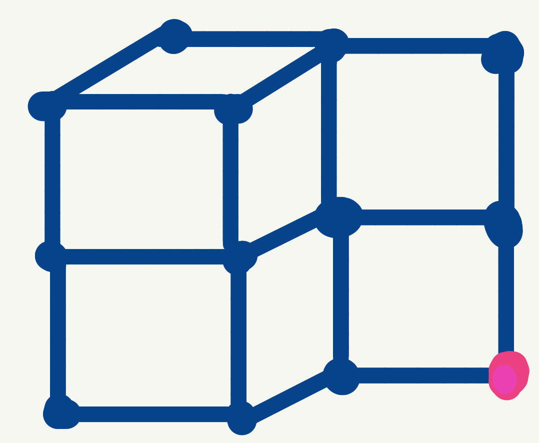


② corner of a face



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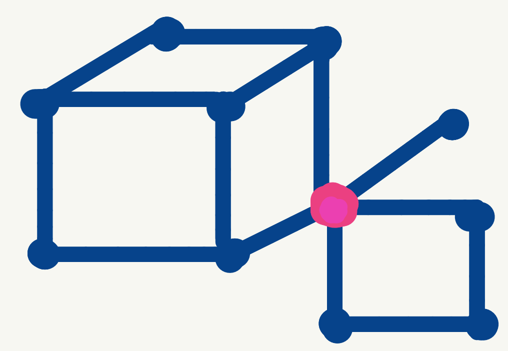

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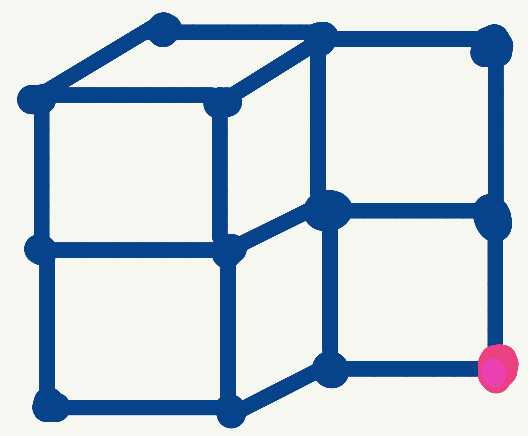
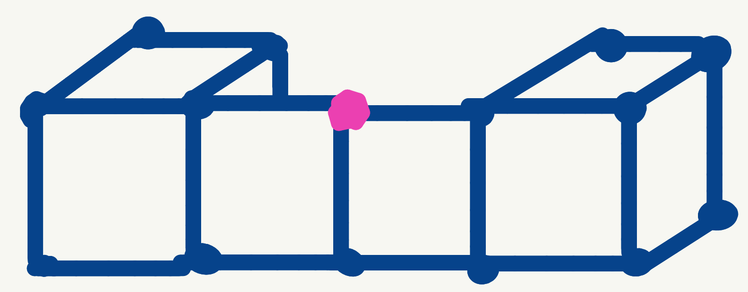


③ Degree 3 vertex not in a cube



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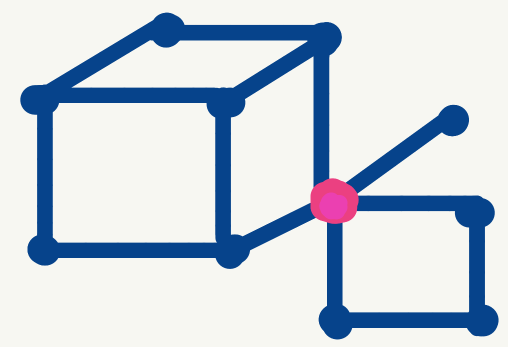

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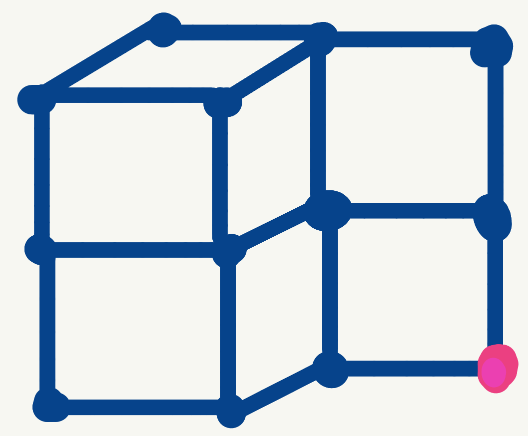

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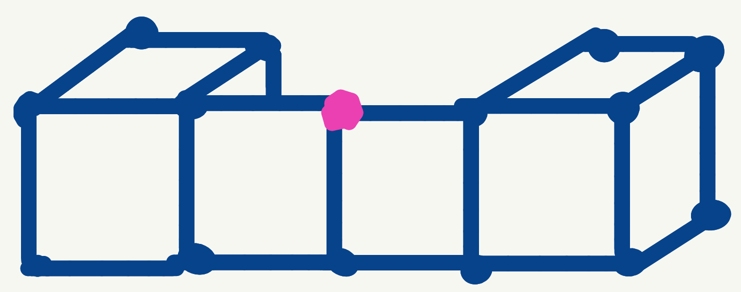

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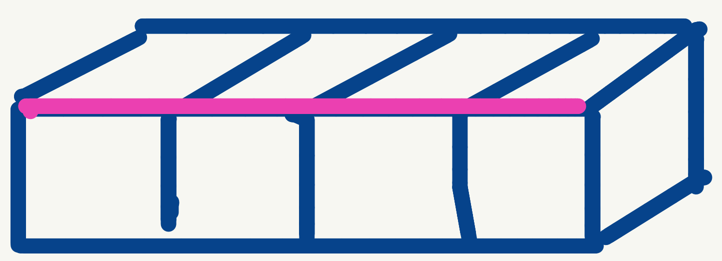

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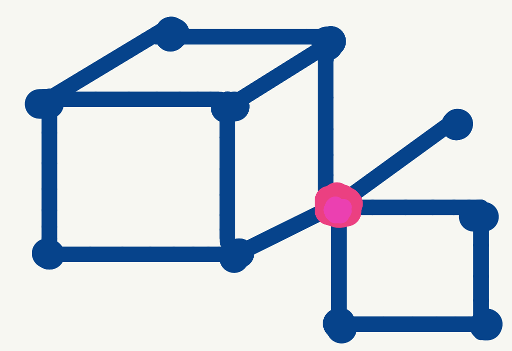

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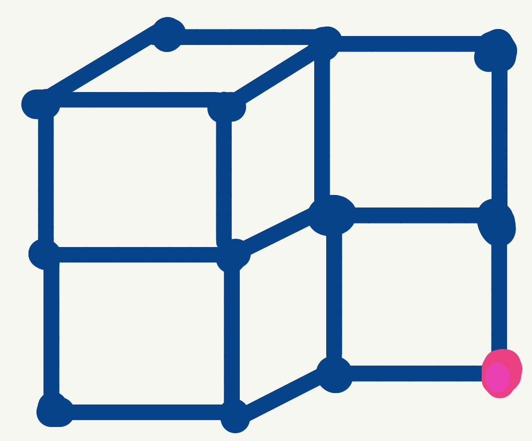

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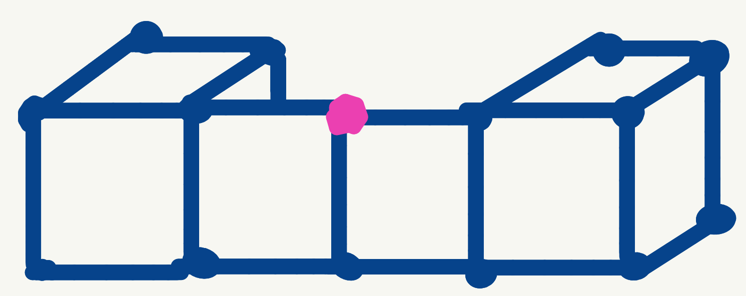

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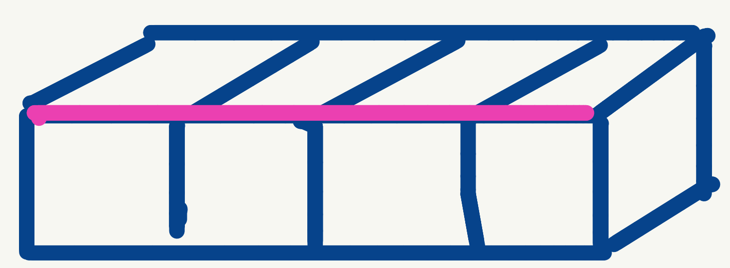

Recognition lemma: Each substructure can be identified from the matrix of boundary distances.

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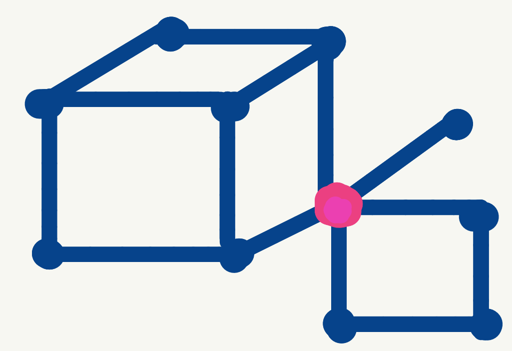

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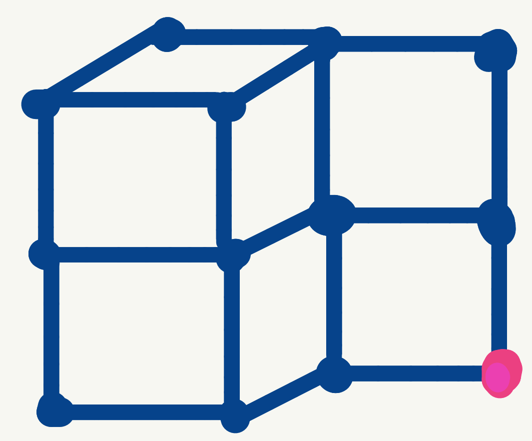

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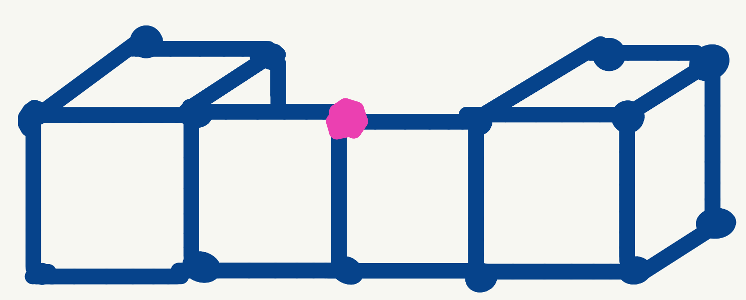

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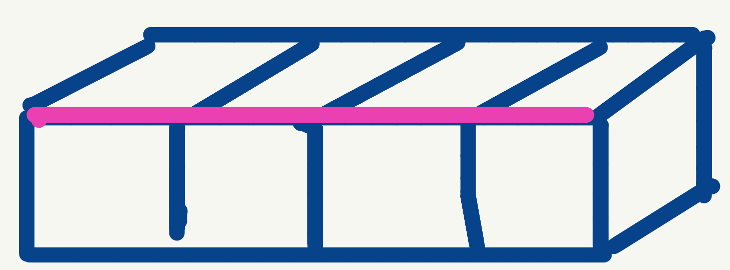

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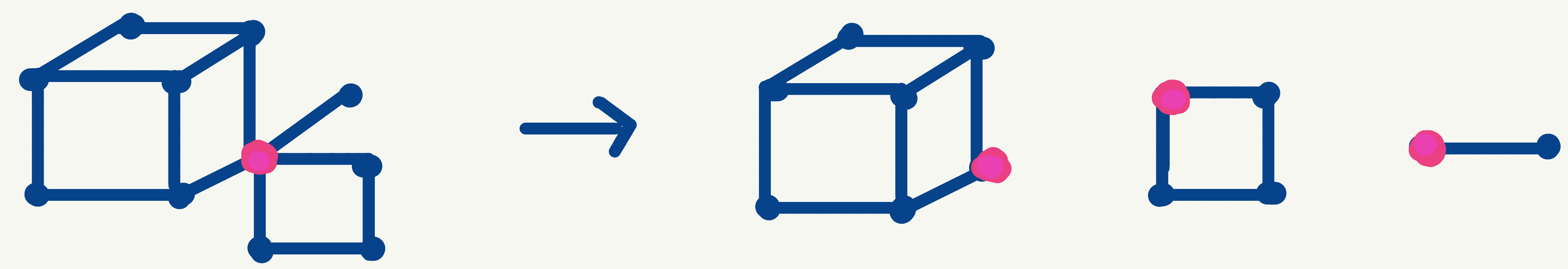

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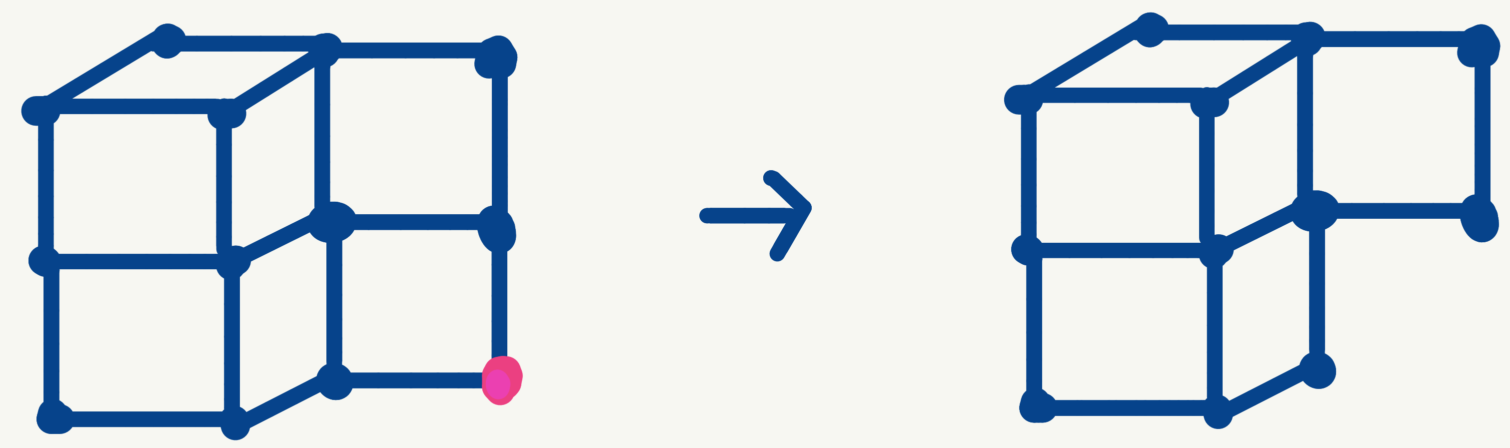

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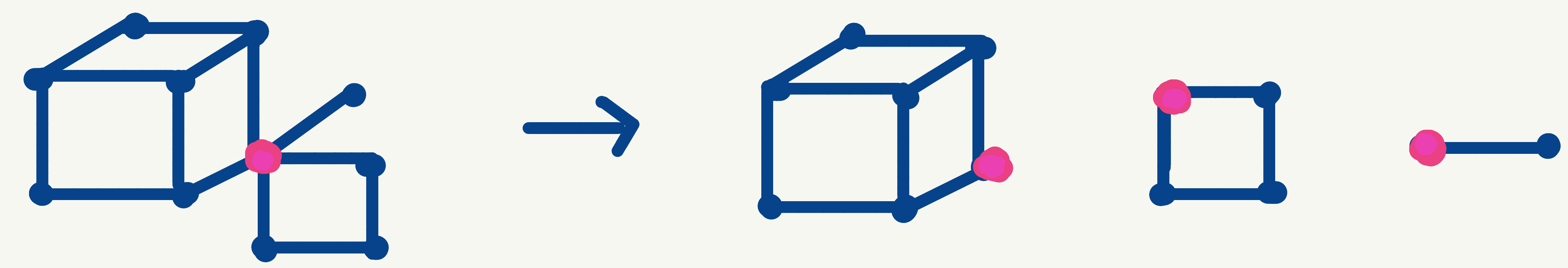

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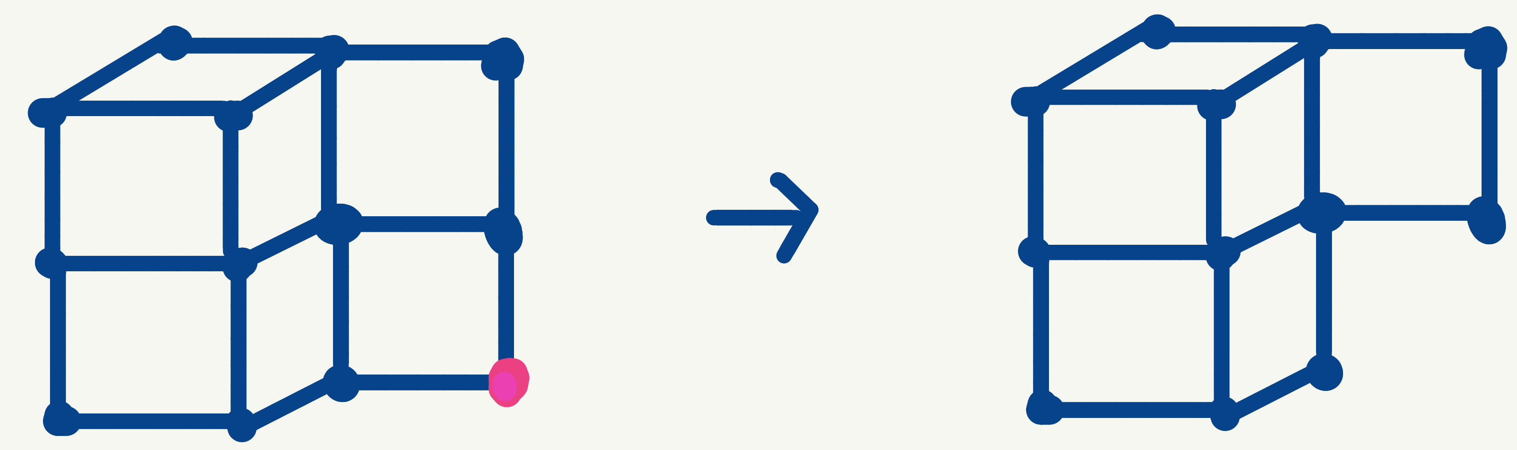

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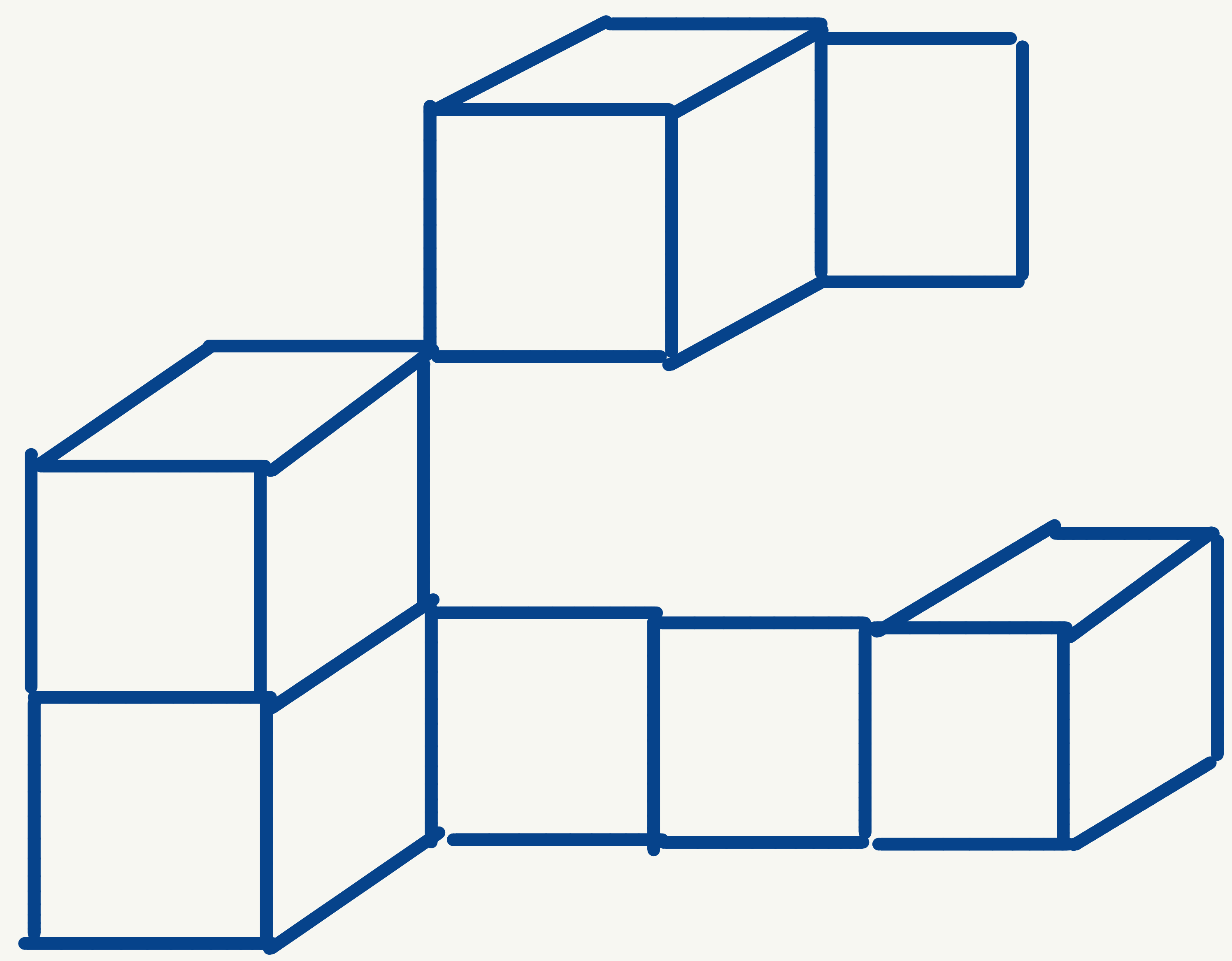
Procedure : While X is not an edge,

Apply ① if possible, then iterate on components.

Else apply ② if possible, then iterate.

Else apply ③ if possible, then iterate.

Else apply ④, then iterate.



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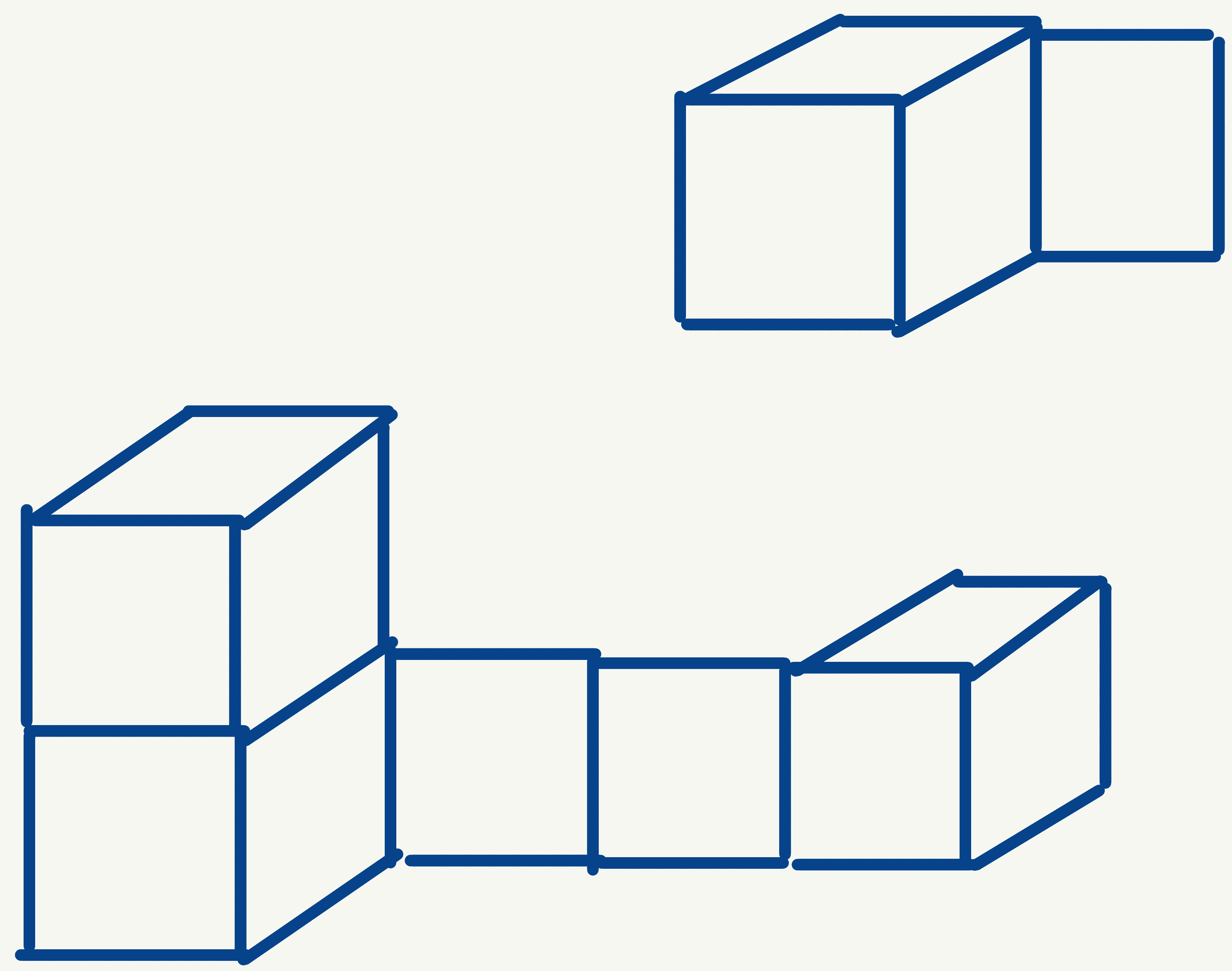

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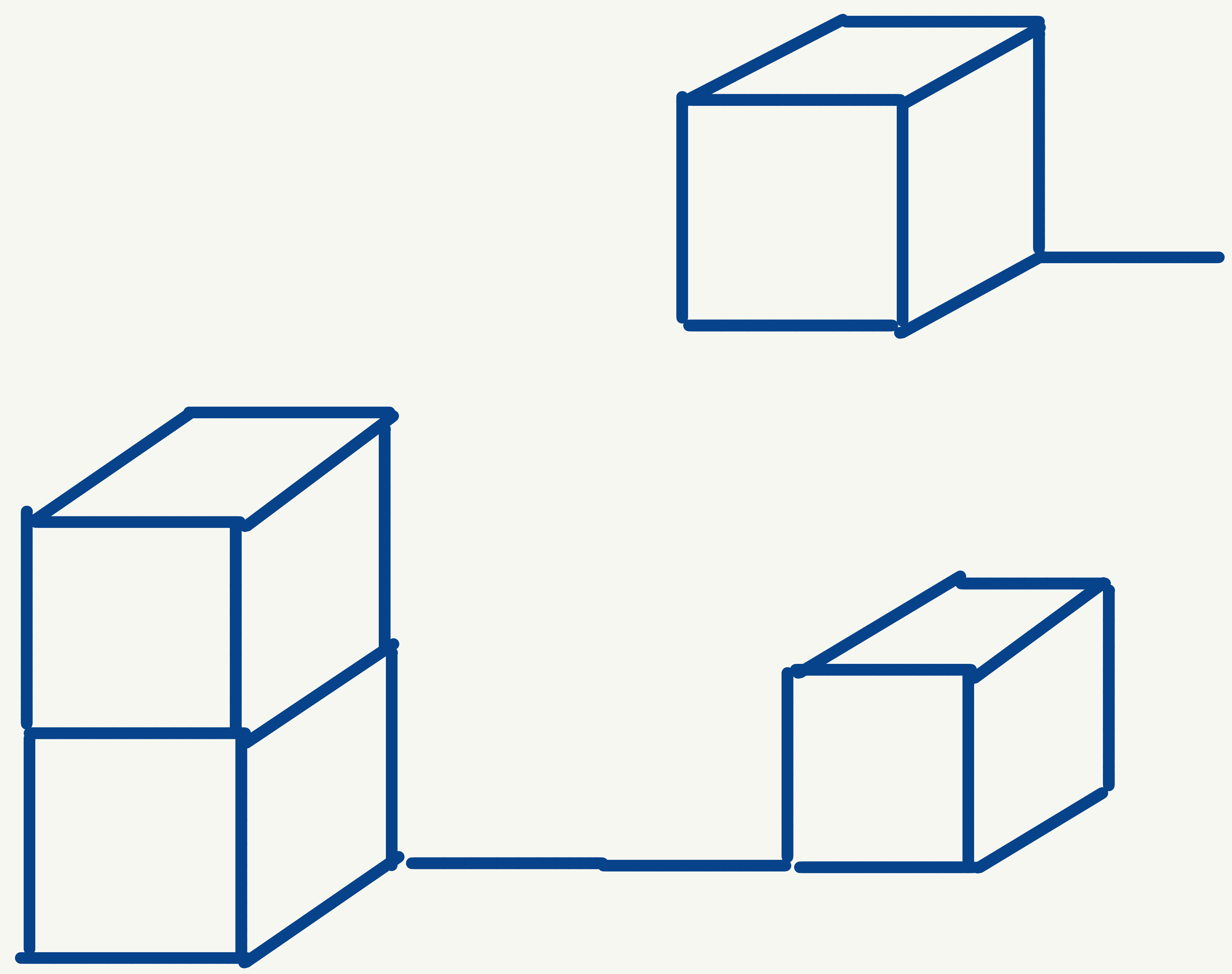

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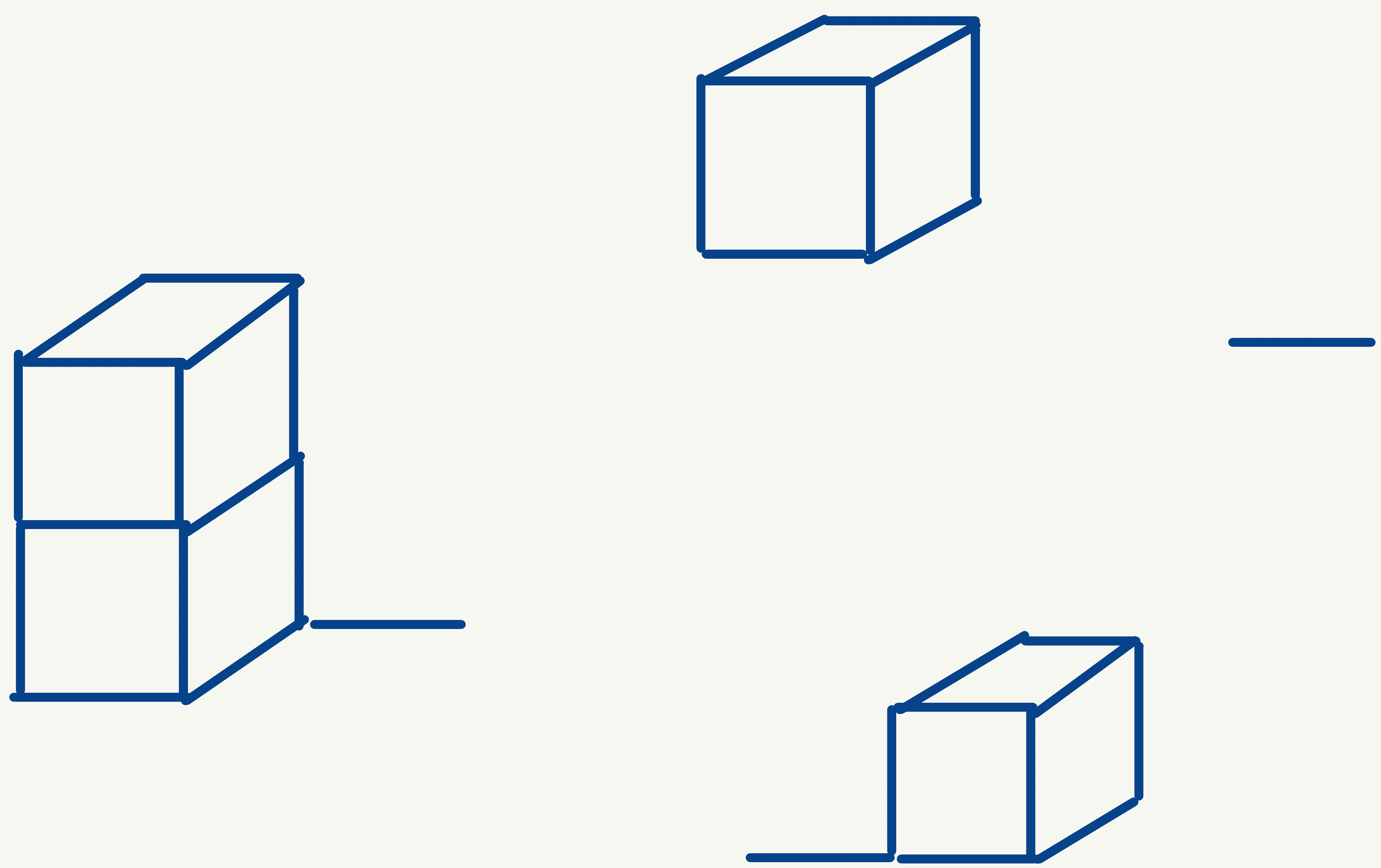

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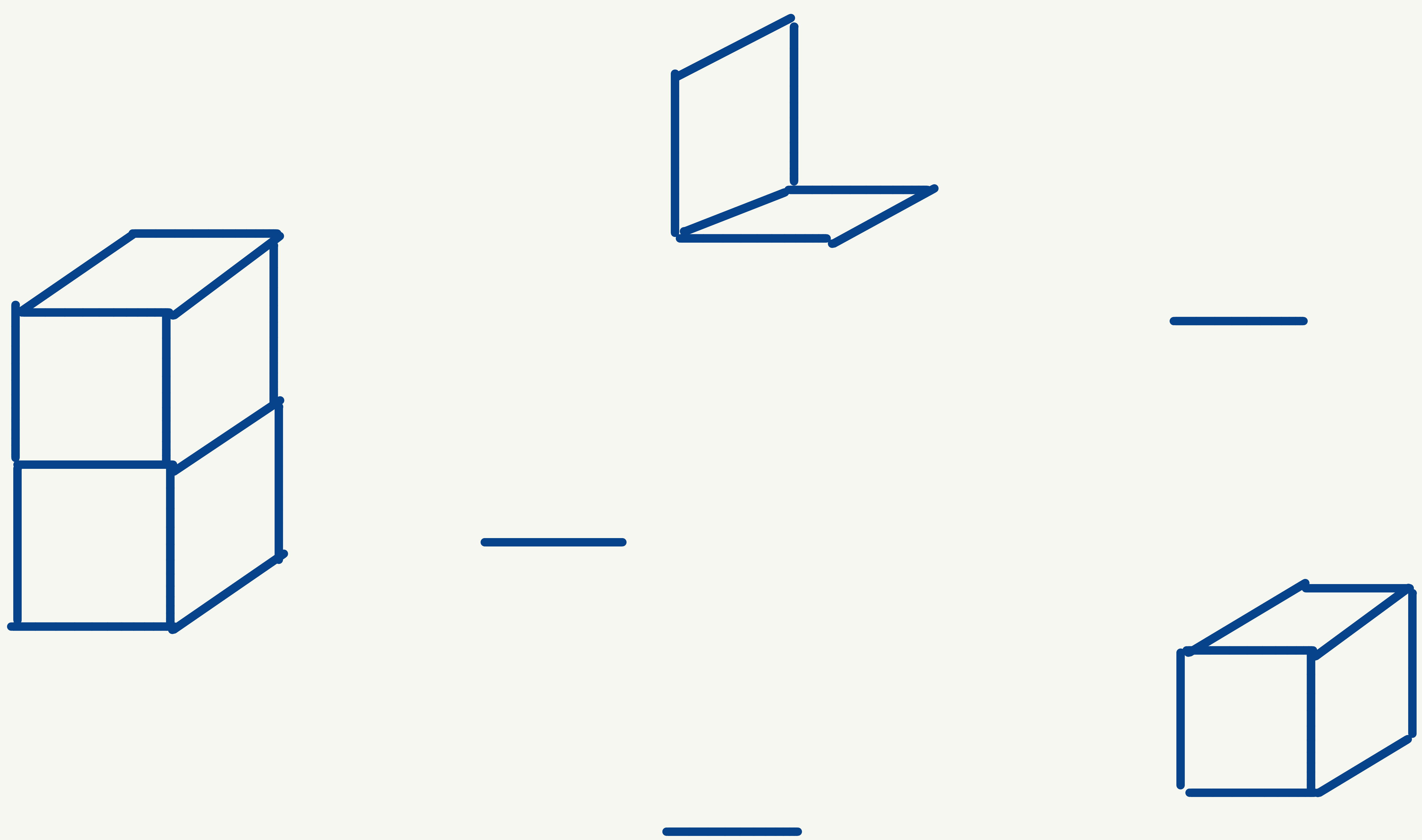

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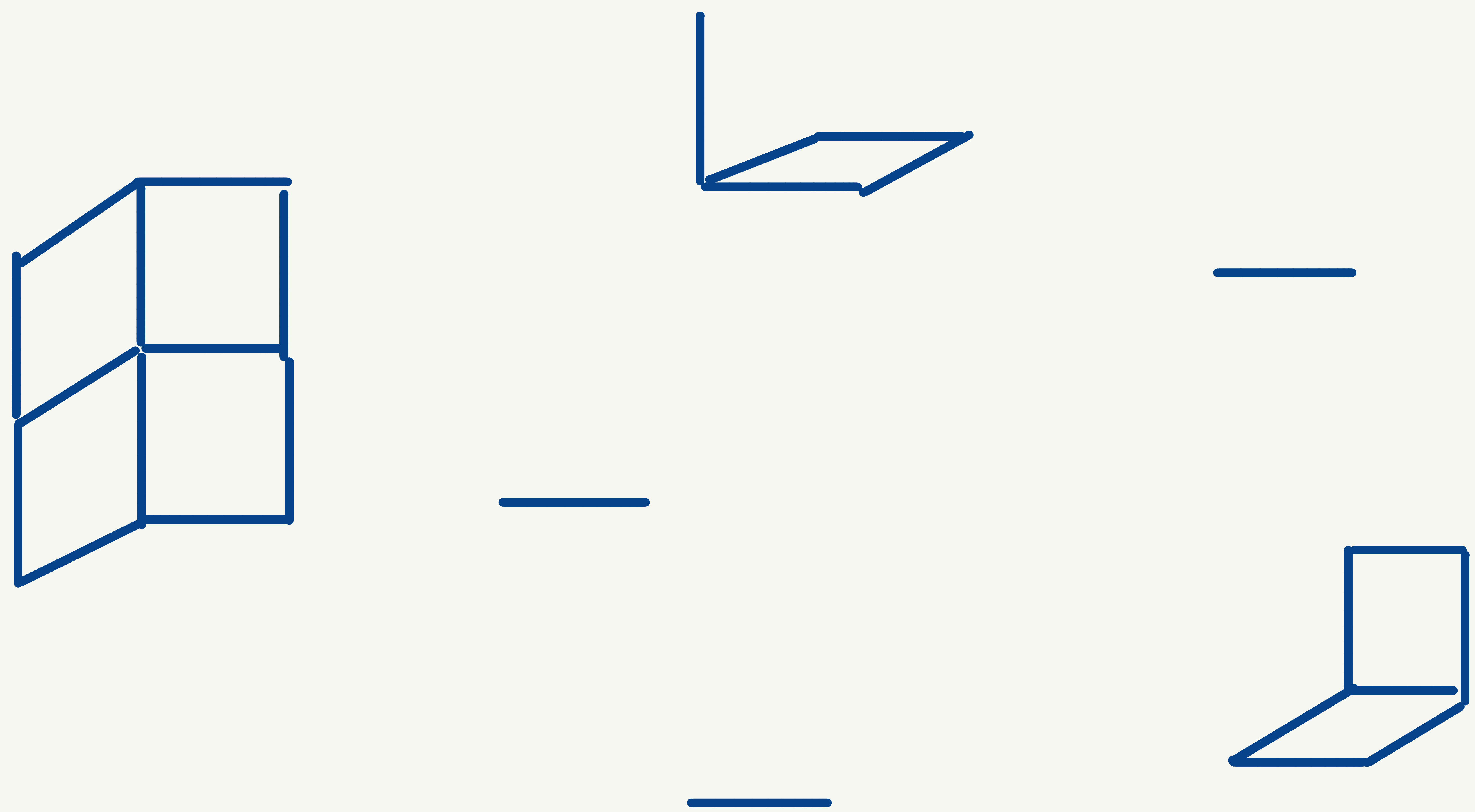

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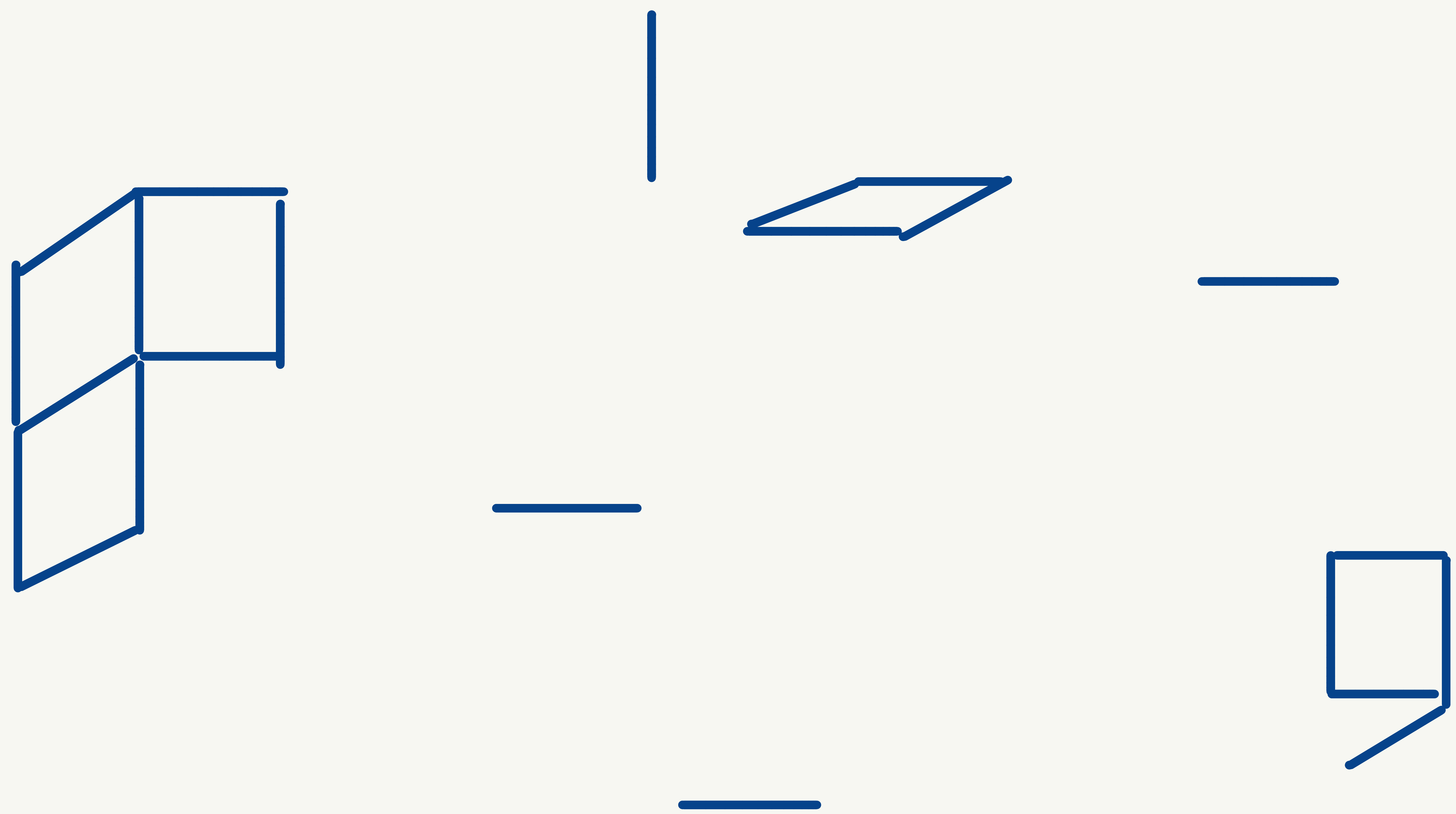

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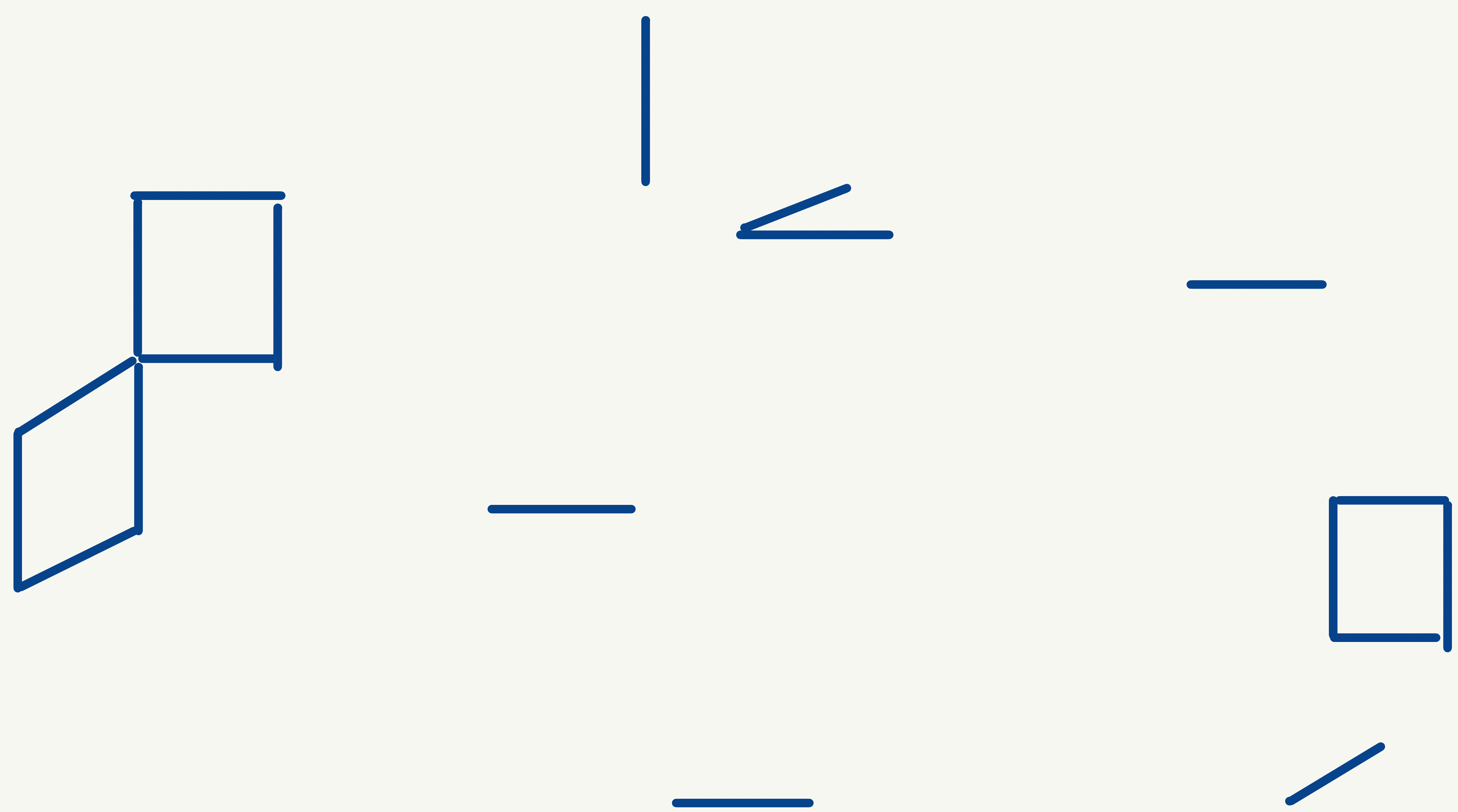

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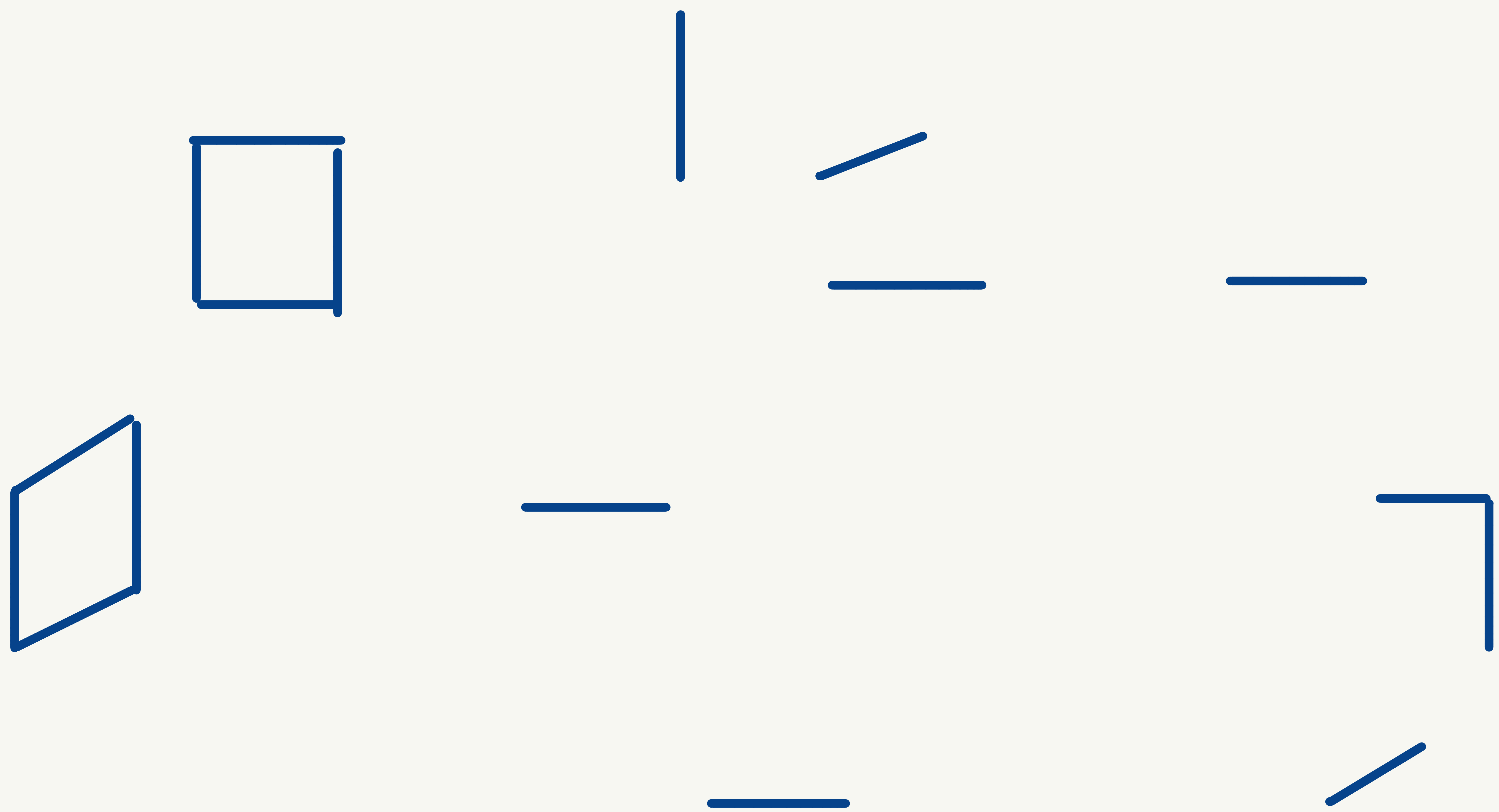

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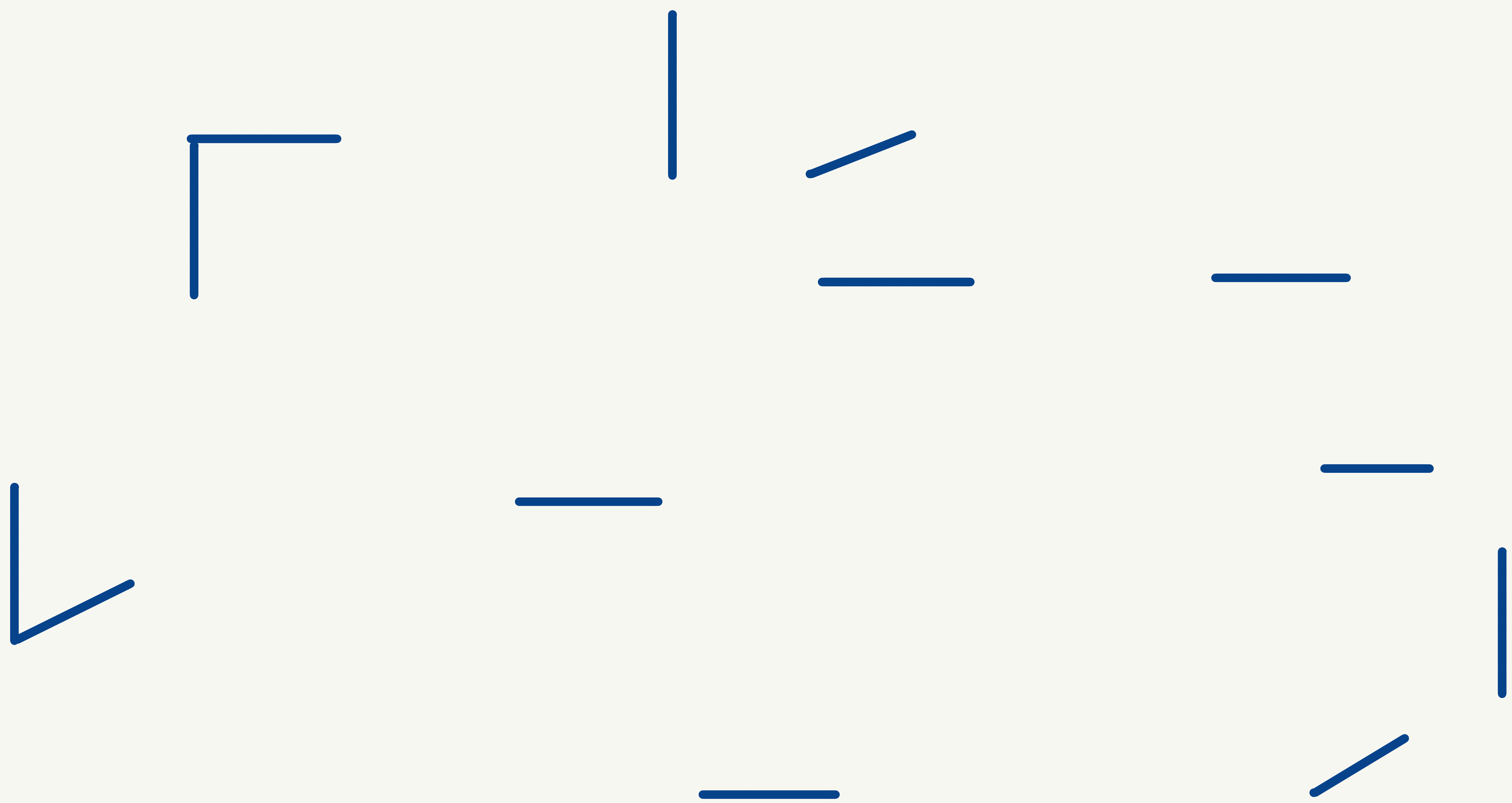

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Structure lemma: Every finite CAT(0) cube complex embeddable in $\mathbb{R}^3$ that is not a single edge contains one of these 4 structures

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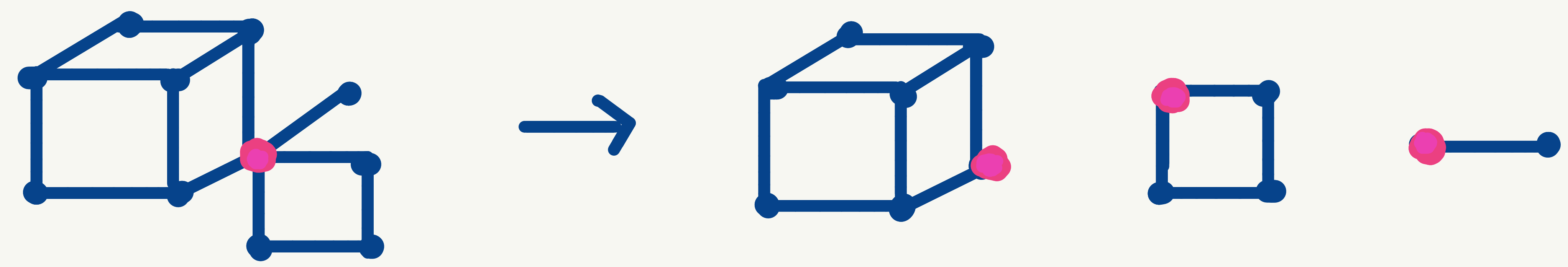

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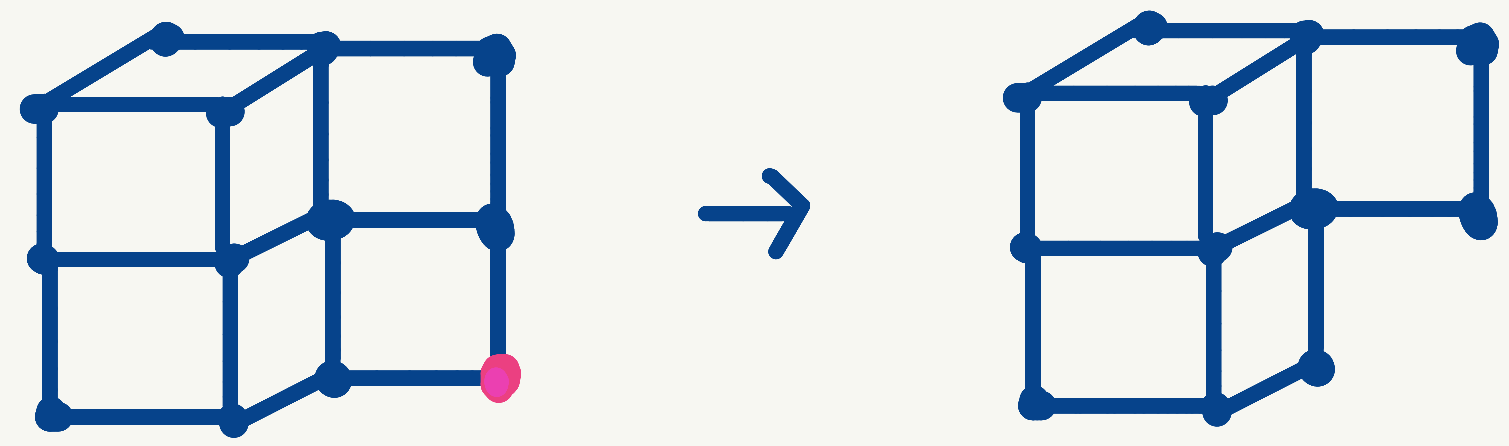

3. Degree 3 vertex not in a cube

4. Row of cubes

Recognition lemma: Each substructure can be identified from the matrix of boundary distances.

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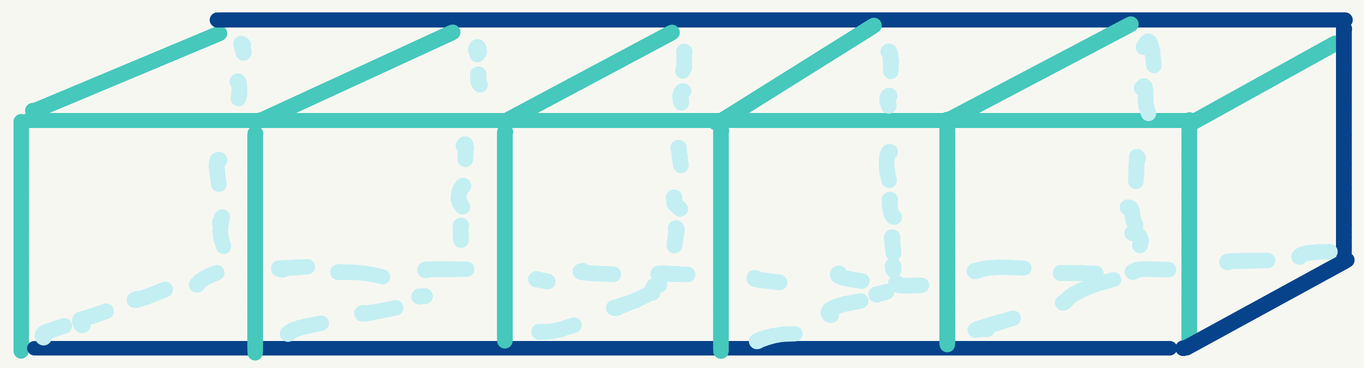
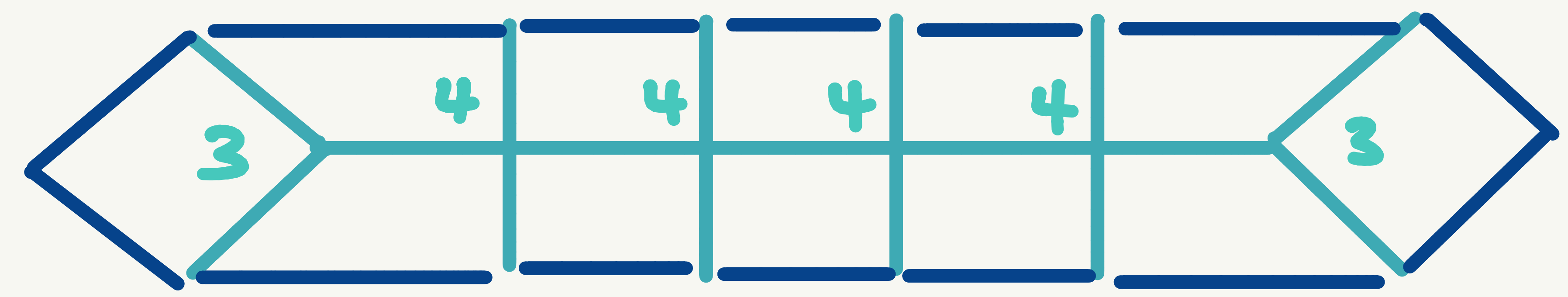
Recognition lemma (for rows of cubes)



Row of cubes

↔



Good row configuration in boundary graph

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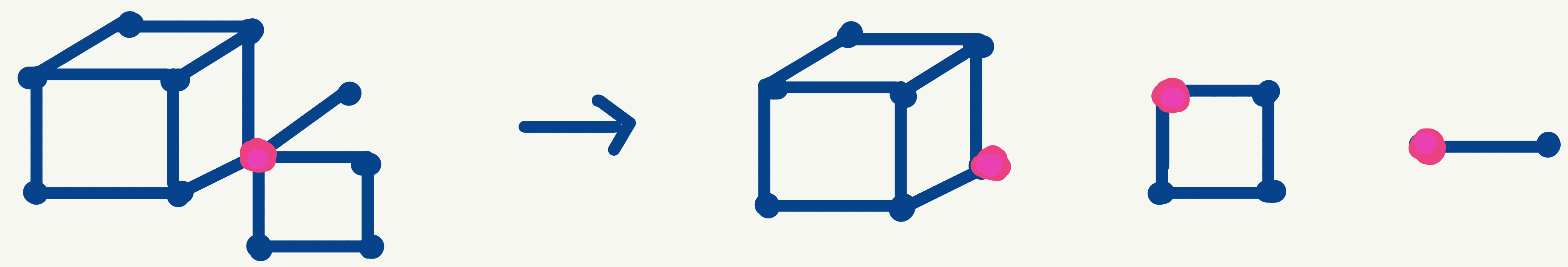

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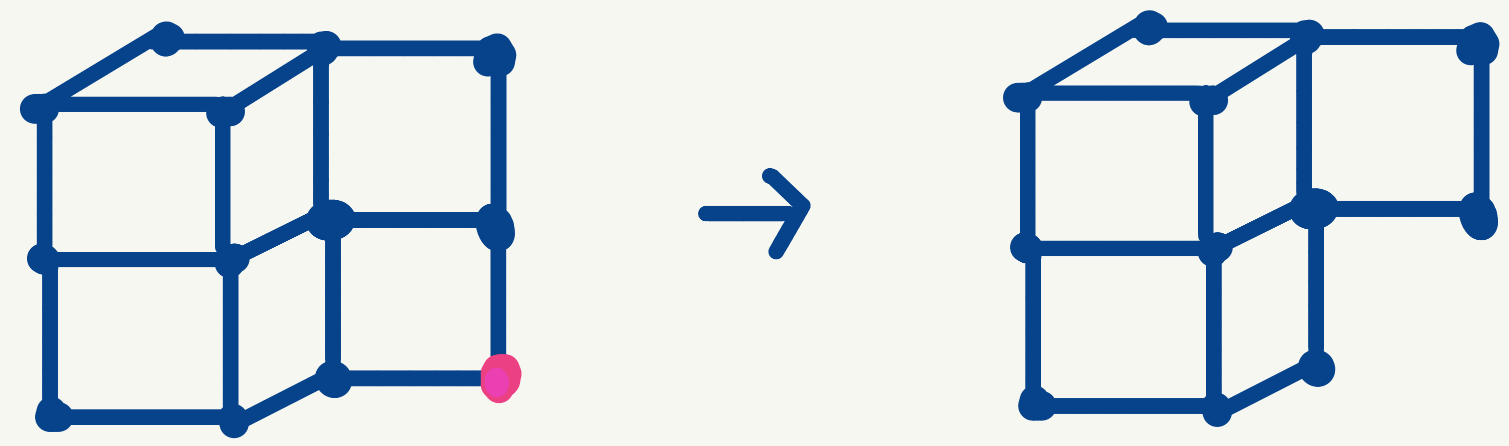

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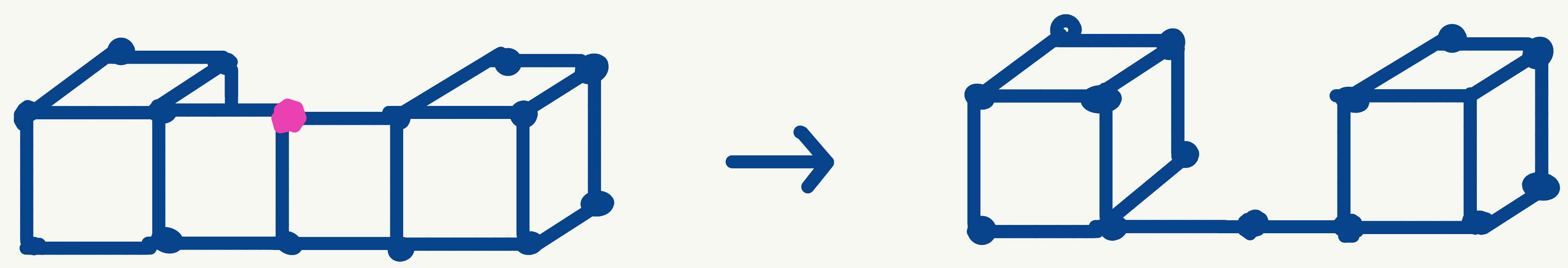

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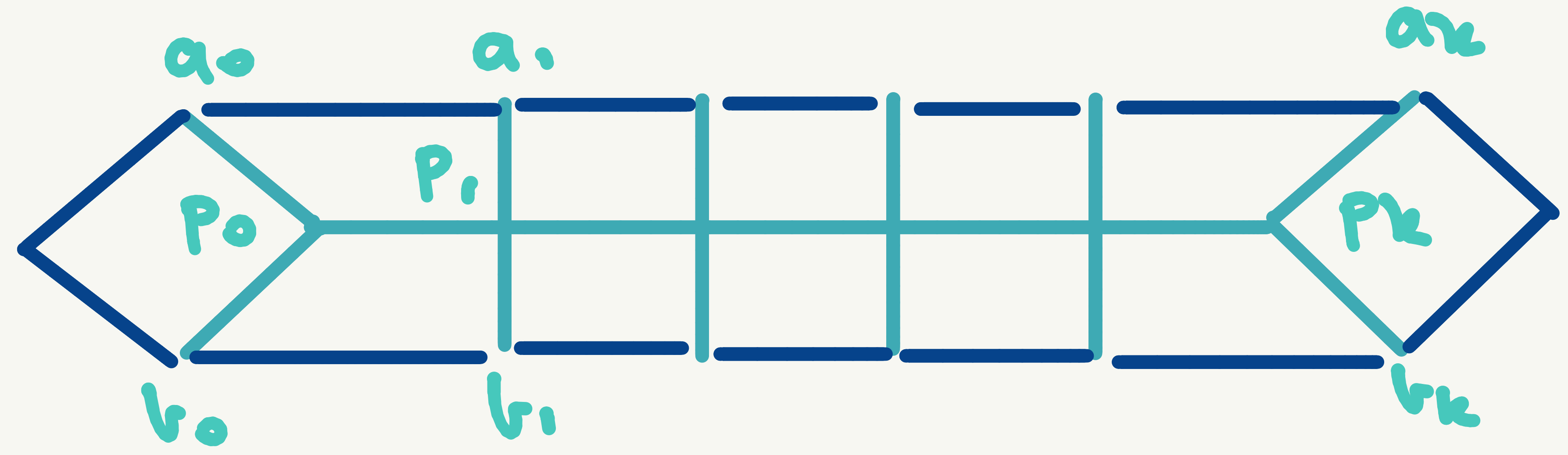
Proof idea: Enough to find a good configuration in ∂X

CASE 1
$X \cong B^3$



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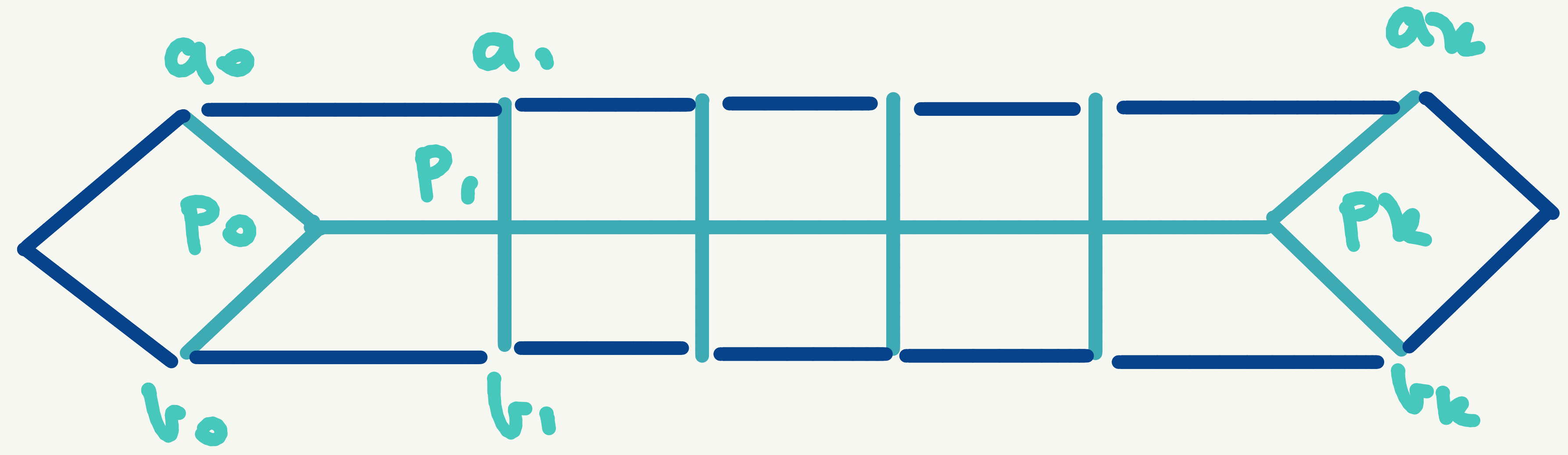

Let's count paths with specific degrees

Path with degrees $3, 4, 4, \ldots, 4, d=3$ or $d \geq 5$

↳ row config

Path with degrees $3, 4, 4, \ldots, 4, 3$

↳ good row config

Path with degrees $3, 4, 4, \ldots, 4, d \geq 5$

↳ bad row config

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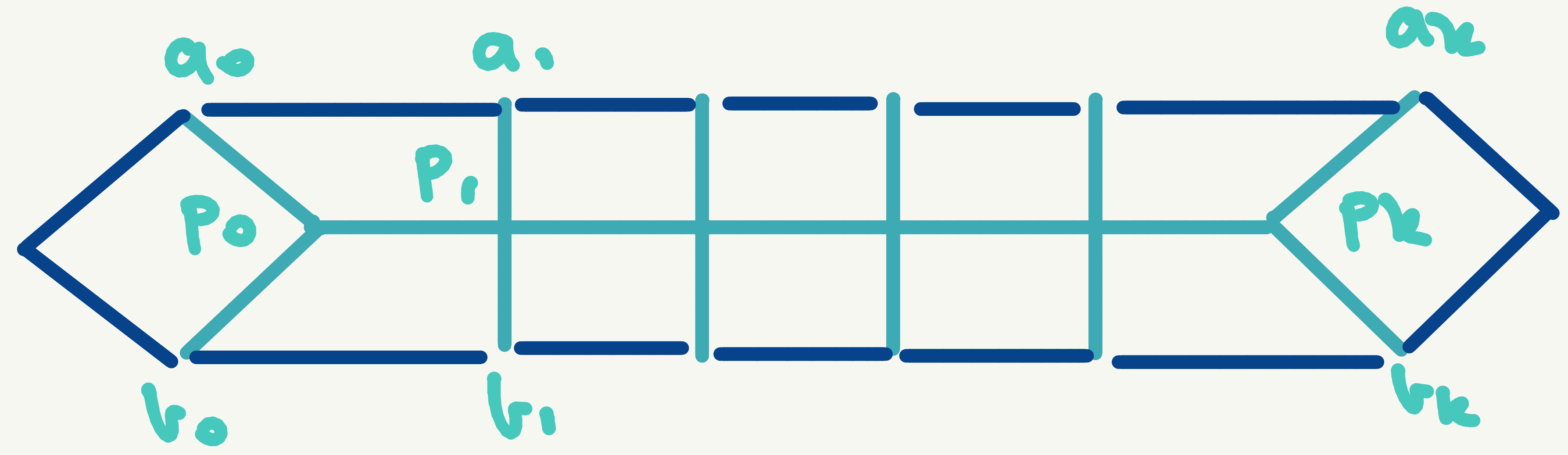
$\boxed{\text{CASE 1} \quad X \cong B^3}$



- Every vertex of deg 3 in $\partial(X)$ is the startpoint of 3 row configurations
- If $x \in \partial X^0$ and $d(x) \geq 5$, then at most $\lfloor \frac{2}{3} d(x) \rfloor$ row configurations end at x.

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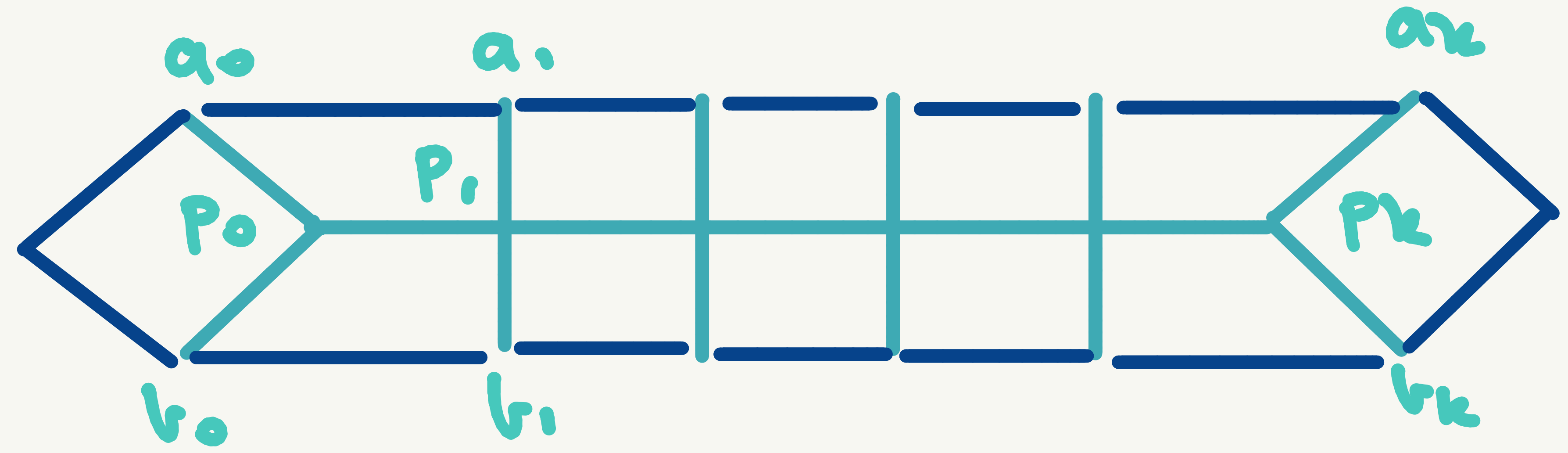
$\boxed{\begin{array}{c}\text{CASE 1}\\ X \cong B^3\end{array}}$



Now $\quad 2 = \sum_{k \geq 3} n_k - |E| + |F| \qquad \partial X \cong S^2, \qquad n_k = \#$ vertices on ∂X with deg k

$$8 = 4 \sum_{k \geq 3} n_k - 4|E| + 4|F|$$

$$8 = 4 \sum_{k \geq 3} n_k - 2|E| \qquad 4|F| = 2|E|$$

$$8 = \sum_{k \geq 3} 4n_k - kn_k \qquad 2|E| = \sum \text{degrees} = \sum_{k \geq 3} kn_k$$

$$n_3 = 8 + \sum_{k \geq 5} (k-4) n_k$$

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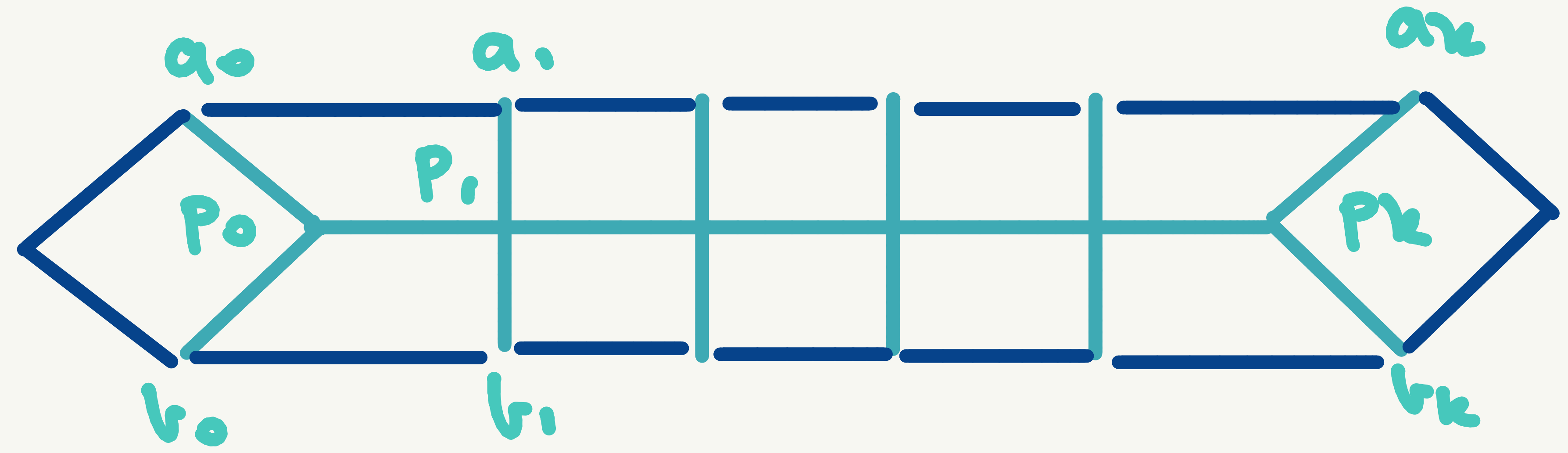
Proof idea: Enough to find a good configuration in ∂X

$\boxed{\text{CASE 1} \quad X \cong B^3}$



Now

$$2 = \sum_{k \geq 3} n_k - |E| + |F|$$

$$8 = 4\sum_{k \geq 3} n_k - 4|E| + 4|F|$$

$$8 = 4\sum_{k \geq 3} n_k - 2|E|$$

$$8 = \sum_{k \geq 3} 4n_k - kn_k$$

$$n_3 = 8 + \sum_{k \geq 5} (k-4) n_k$$

Then # good row configs

$= \#$ all row config $- \#$ bad

$$\geq 3n_3 - \sum_{k \geq 5} \left\lfloor \frac{2k}{3} \right\rfloor n_k$$

$$= 24 + \sum_{k \geq 5} \underbrace{\left(3k - \left\lfloor \frac{2k}{3} \right\rfloor - 12\right)}_{\geq 0 \text{ for } k \geq 5} n_k$$

$$\geq 24.$$

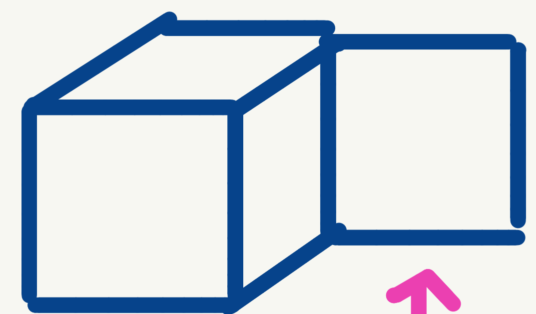
$\square$

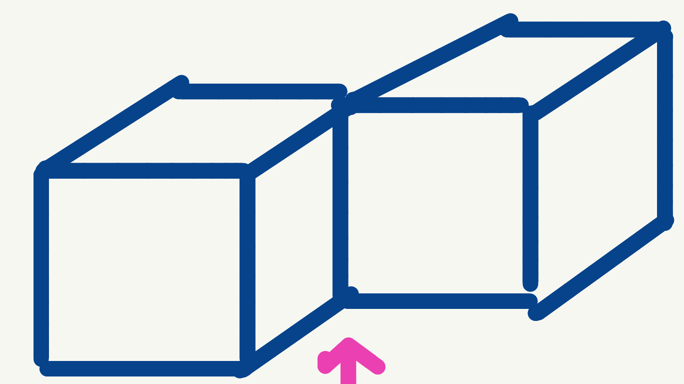
Structure lemma: If X does not contain 1, 2, 3 then X has a row of cubes.

Proof idea:

$\boxed{\text{CASE 2} \quad X \not\cong B^3}$

obstructions to being $\cong B^3$:





Structure lemma: If X does not contain ①, ②, ③ then X has a row of cubes.

Proof idea:

$\boxed{\begin{array}{c}\text{CASE 2} \\ X \not\cong B^3\end{array}}$

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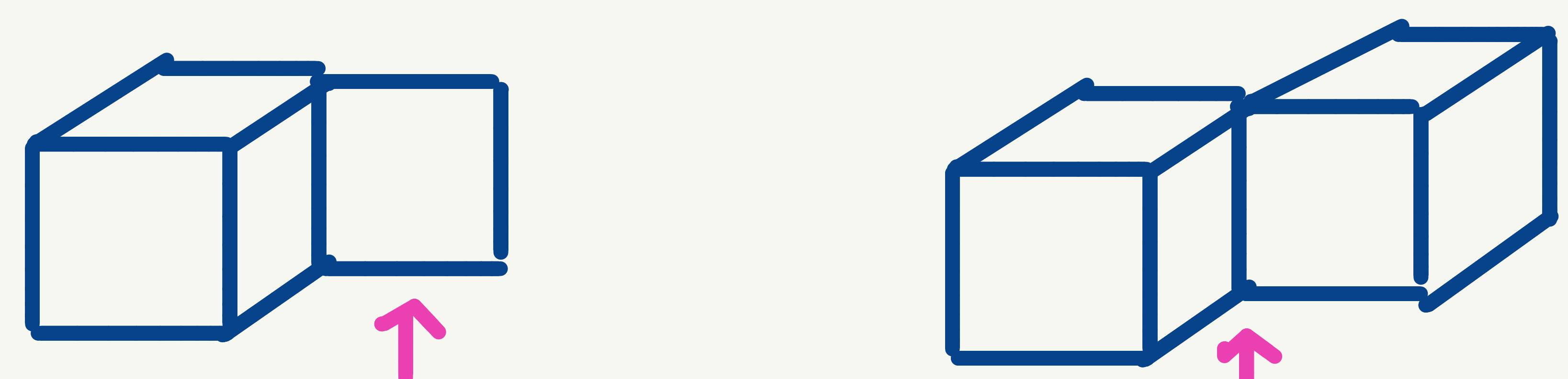
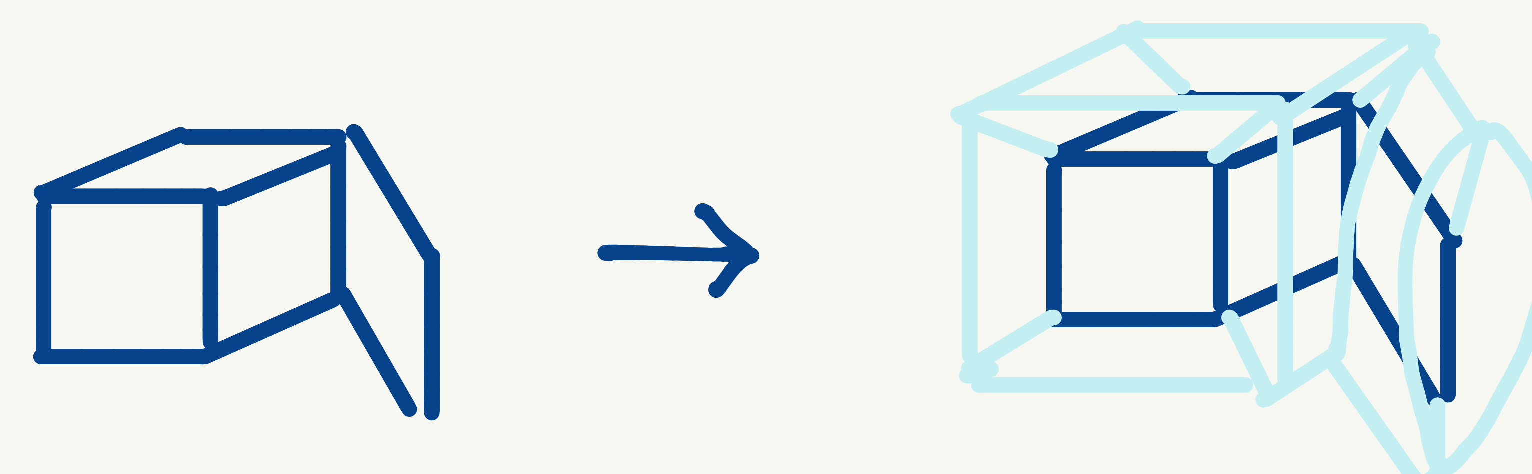


free face   cubes glued along edge

Construct a 'thickened' complex $\mathbb{X}$ from X :



X $\mathbb{X}$

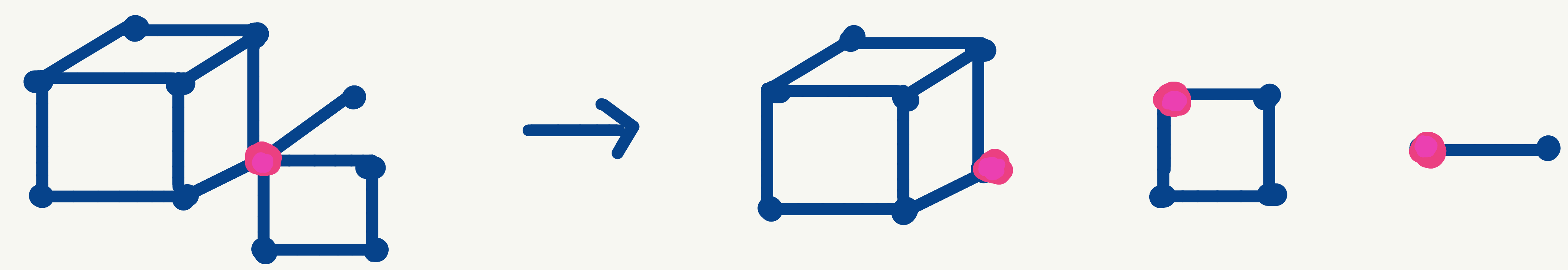
- $\mathbb{X}$ is a connected contractible pure 3D cube complex
- $\partial\mathbb{X}$ is homeomorphic to S^2
- there is a 1-1 correspondence between row configurations in ∂X and row configurations in $\partial\mathbb{X}$.

Finish using Case 1.

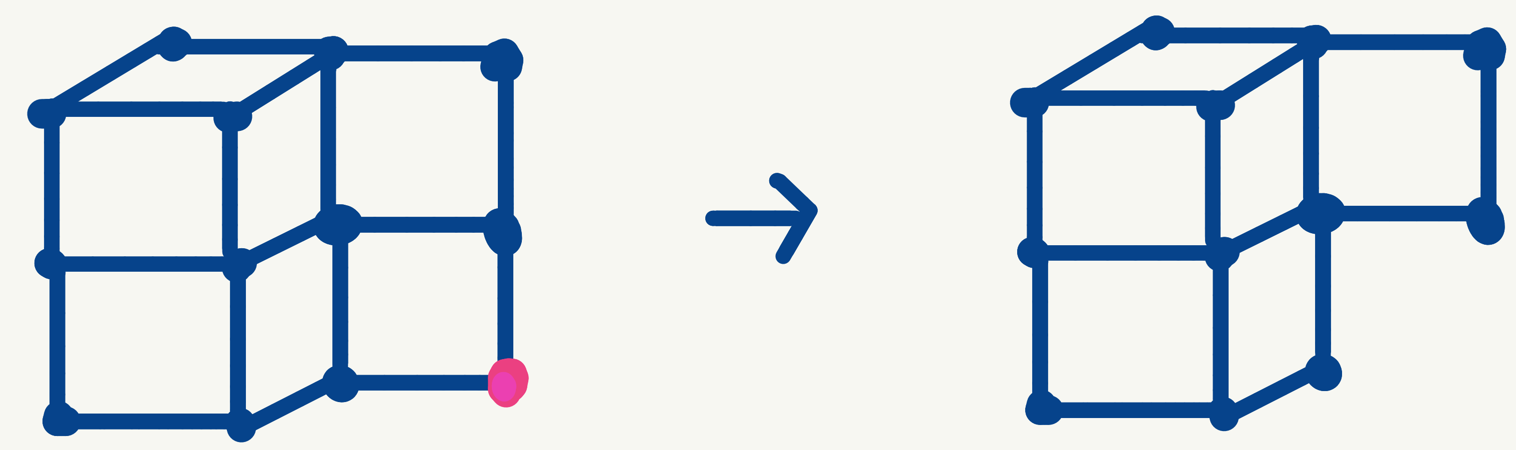
Key Lemmas & Outline

Structure lemma: Every finite CAT(0) cube complex embeddable in $\mathbb{R}^3$ that is not a single edge contains one of these 4 structures

① Cut-vertices



② corner of a face



③ Degree 3 vertex not in a cube

④ Row of cubes

Recognition lemma: Each substructure can be identified from the matrix of boundary distances.

Removal lemma: Each reduction operation leaves a CAT(0) cube complex (contractible + flag) for which we can recover the matrix of boundary distances.

Removal lemma (for rows of cubes)

Sp R is a row of cubes in X. Then

- $X-R$ is simply connected
- $X-R$ is flag

$\Big\}$ CAT(0)

- we can recover the matrix of boundary distances for $X-R$

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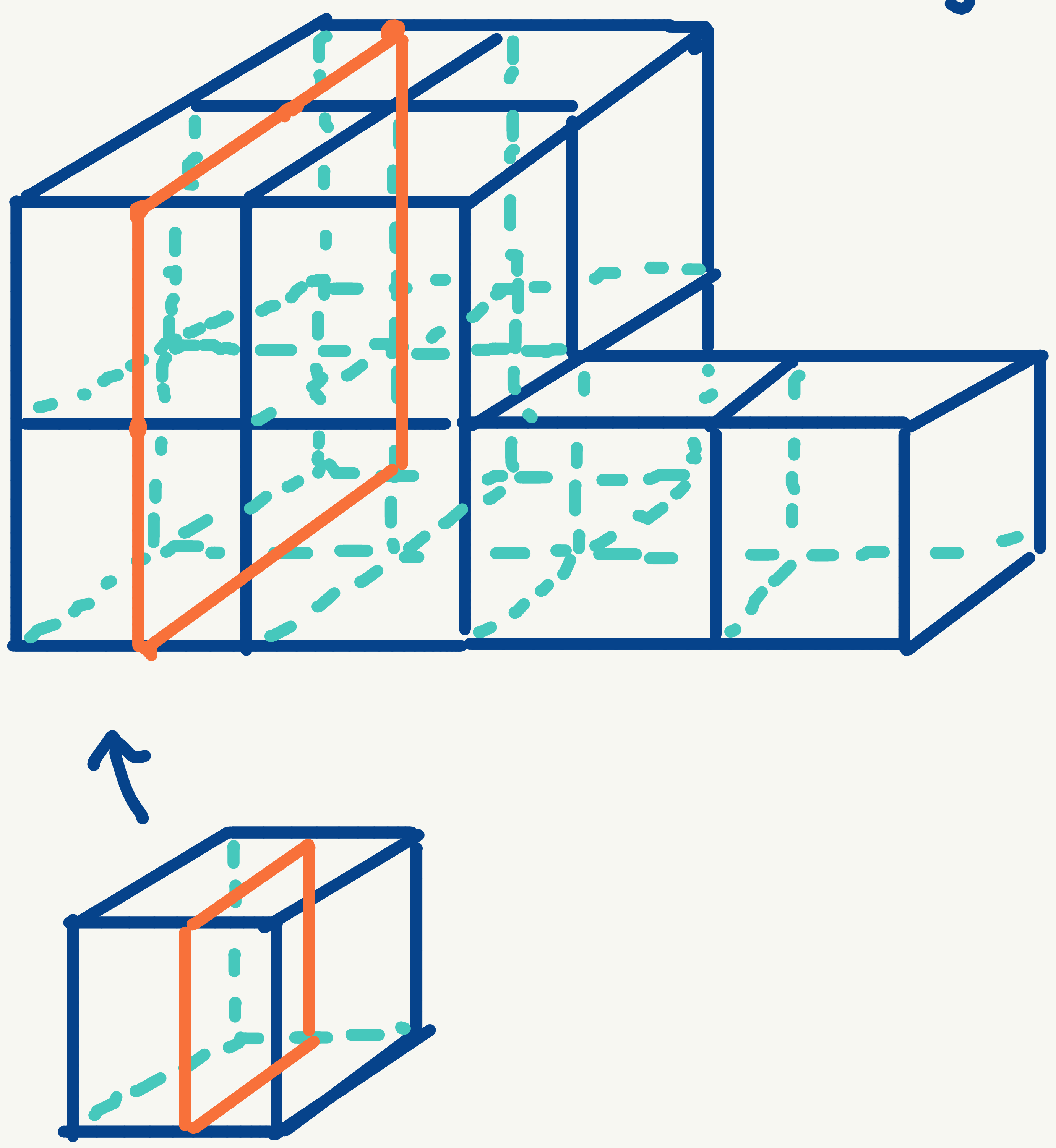
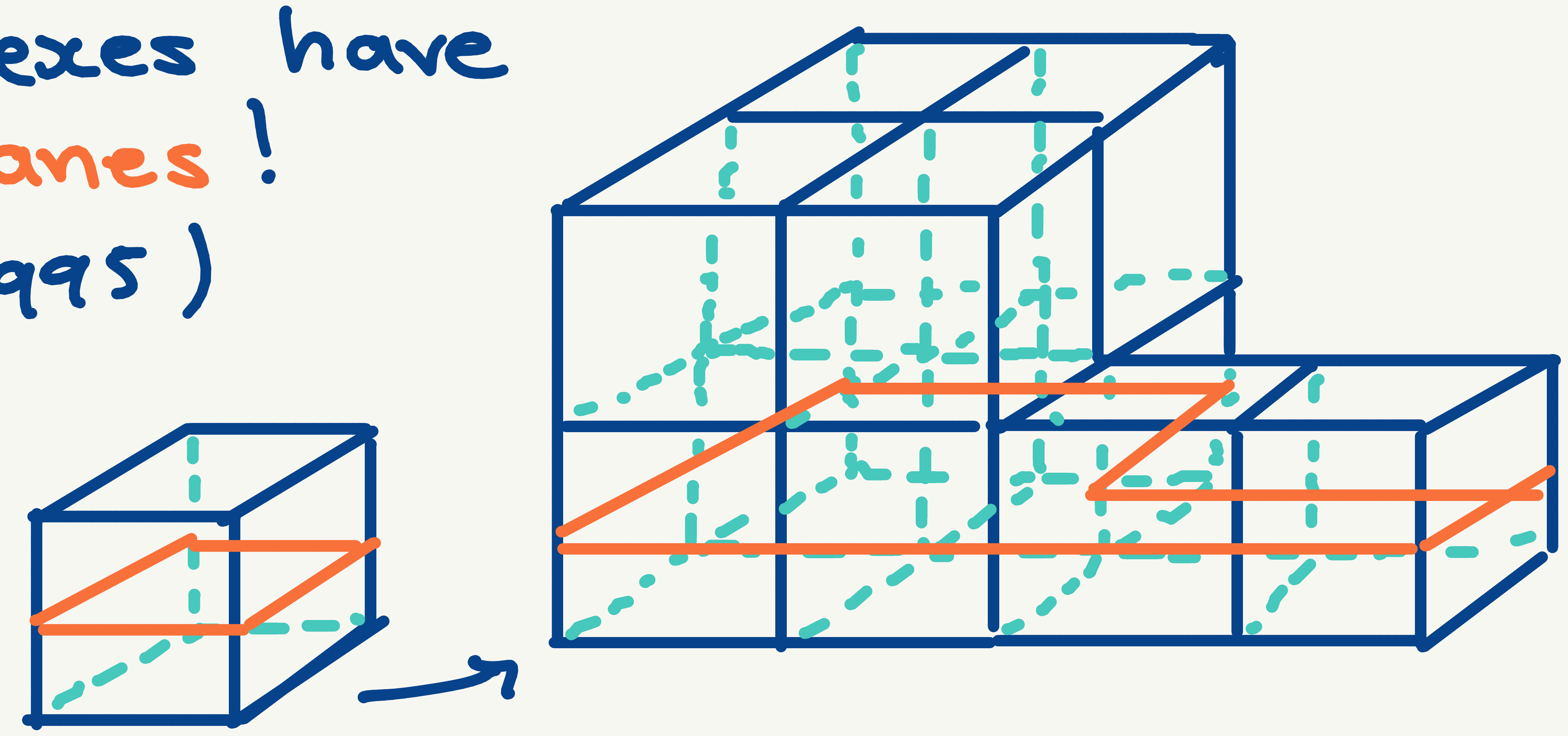
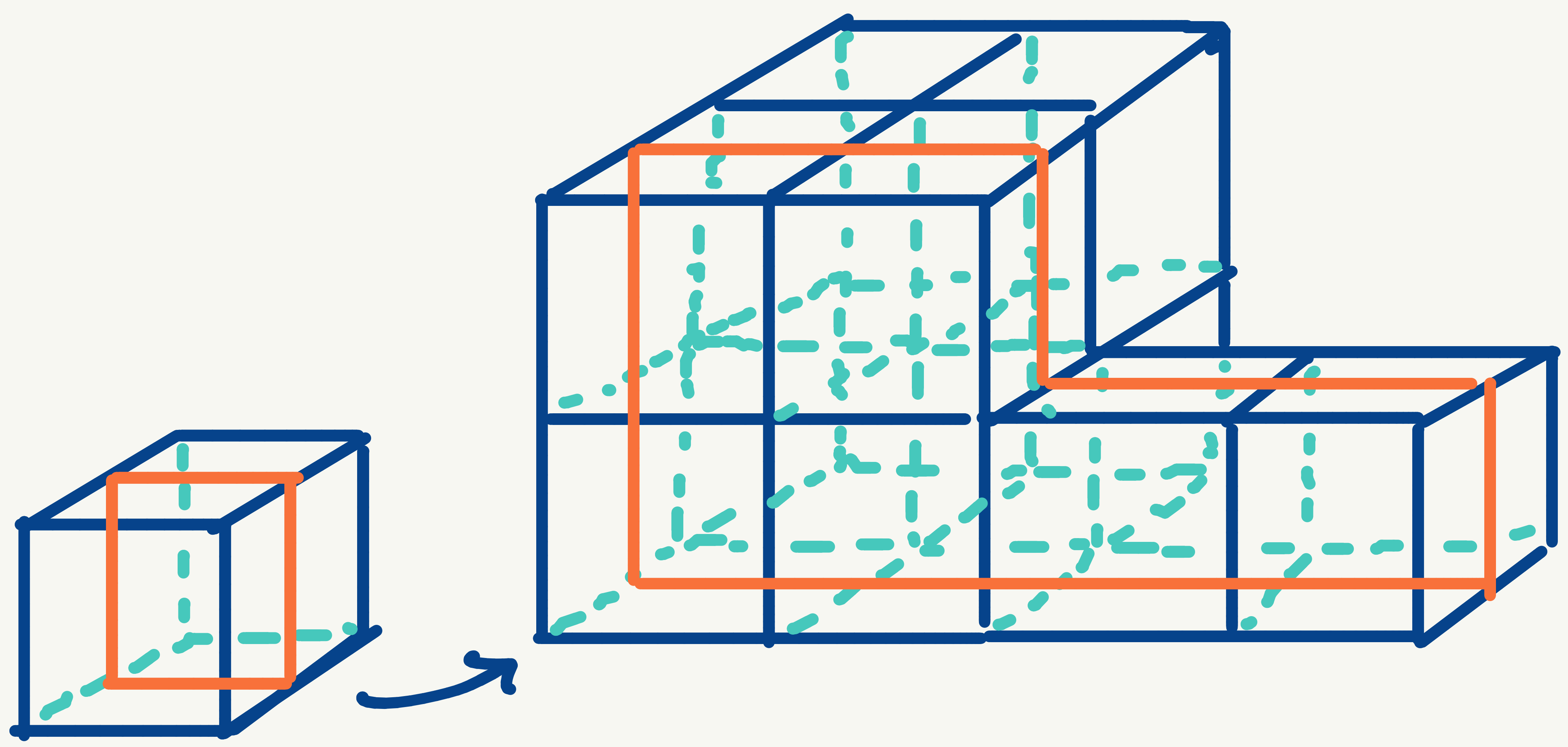
$\left.\begin{array}{l}\\ \\ \end{array}\right\}$ CAT(0)

- we can recover the matrix of boundary distances for $X - R$

Proof: CAT(0) cube complexes have nice hyperplanes! (Sageev 1995)







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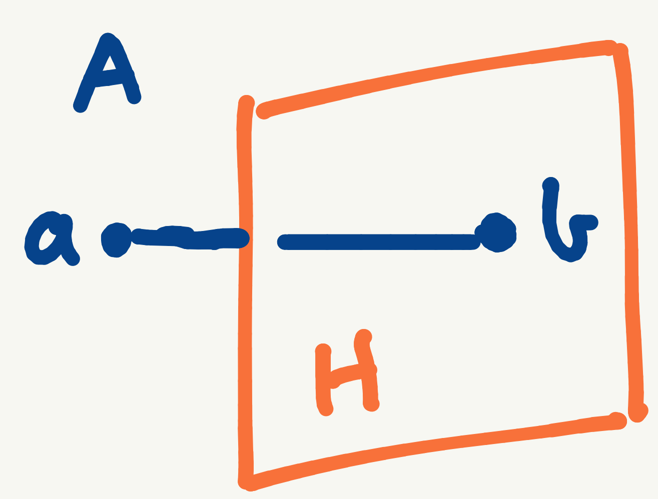
Key lemma: Sp



in X. Then $\forall\, v \in X$,

$$d_X(v, a) = d_X(v, b) + 1 \iff v \text{ is on the side of } H \text{ containing } b.$$

Removal lemma (for rows of cubes)

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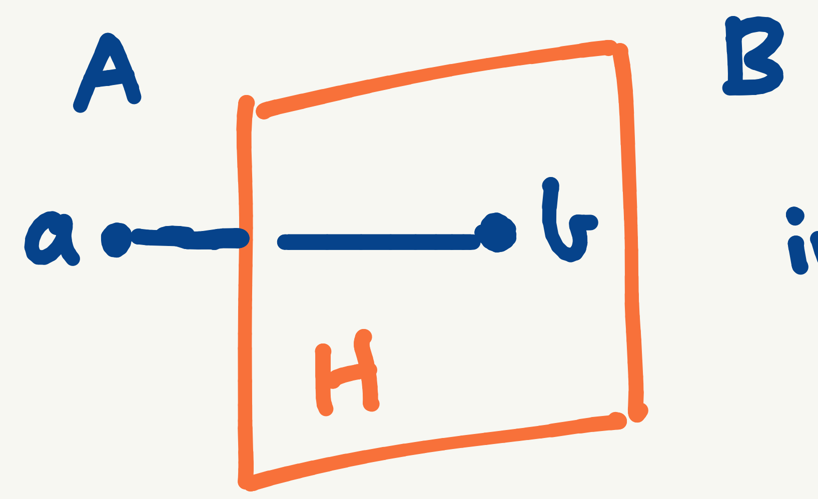
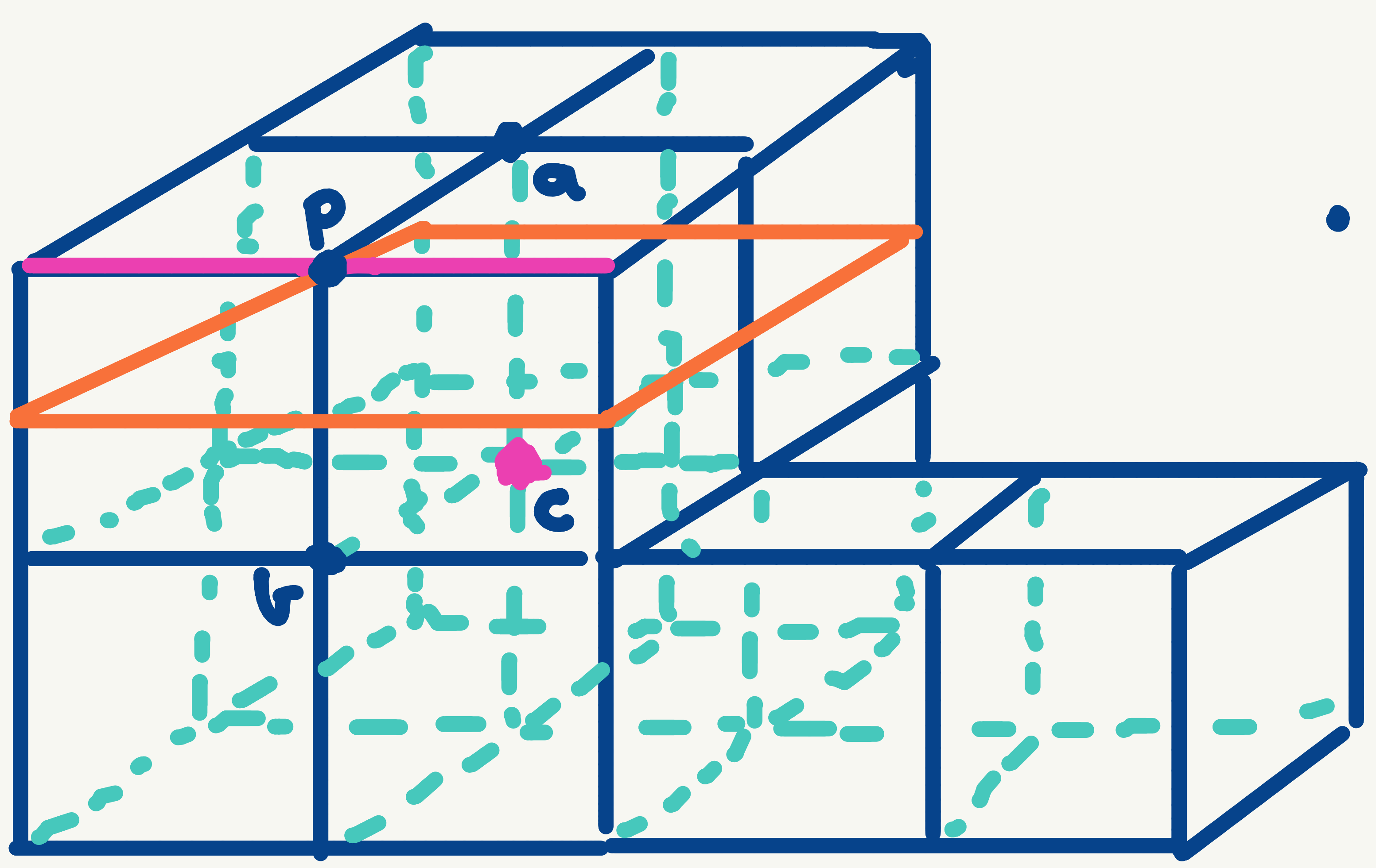


in X. Then $\forall v \in X$,

$$d_X(v, a) = d_X(v, b) + 1 \iff v \text{ is on the side of } H \text{ containing } b.$$

This means:

- If $a, b \in \partial X$, we can determine which side of H any point of ∂X lies.
- If $a \in \partial X$ and we know which side of H v lies, then we can determine $d_X(v, b)$.



In higher dimensions

Conjecture: All finite CAT(0) cube complexes can be reconstructed up to combinatorial type from the matrix of boundary distances.

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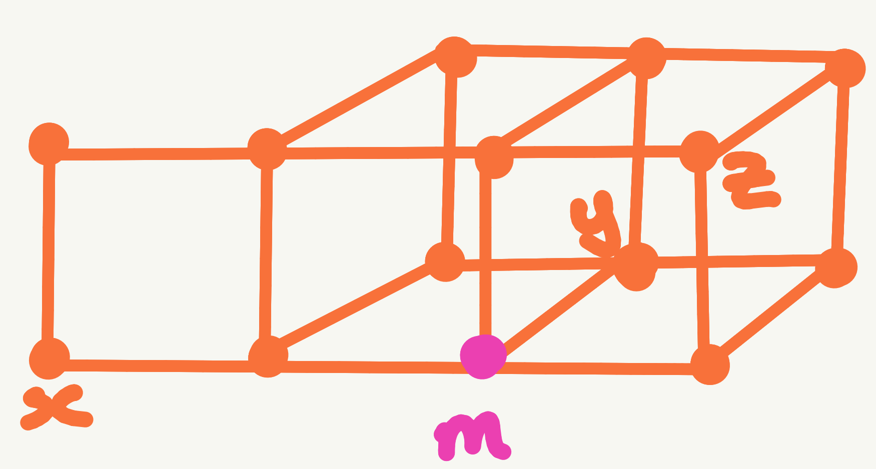
Proof ingredients:

- 1-skeleta of CAT(0) cube complexes are median graphs (Gerasimov 1998, Roller 1998, Chepoi 2000)



A graph is median if any 3 vertices x, y, z have a unique median vertex m on shortest paths between any 2 of x, y, z.

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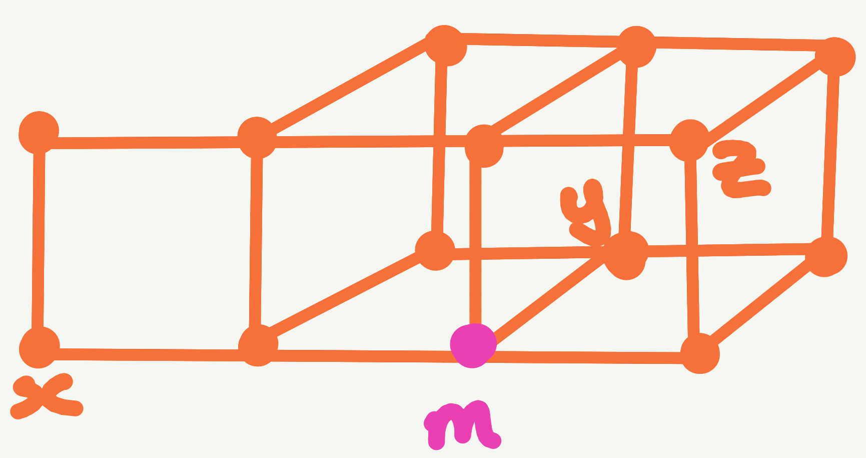

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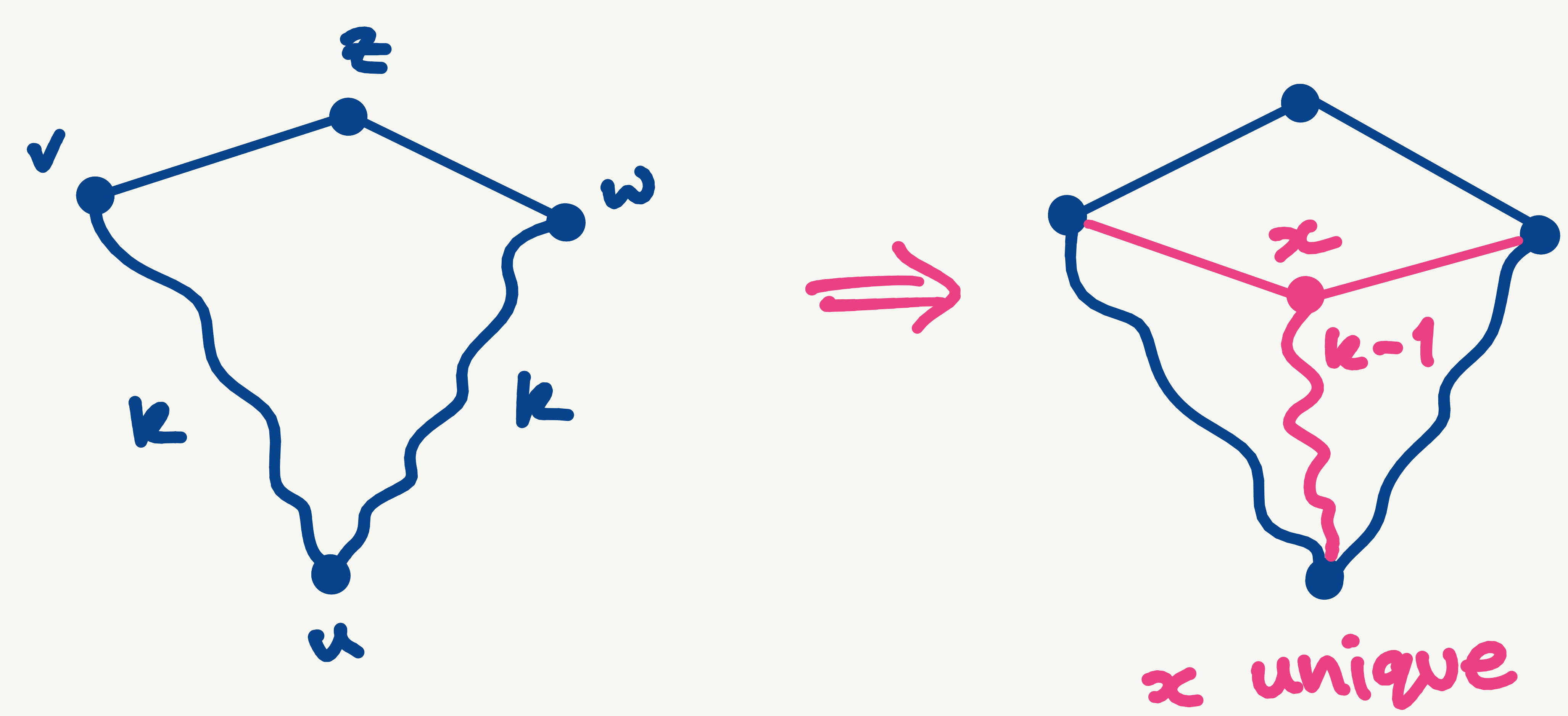
- Quadrangle condition:

  (in a finite median graph)



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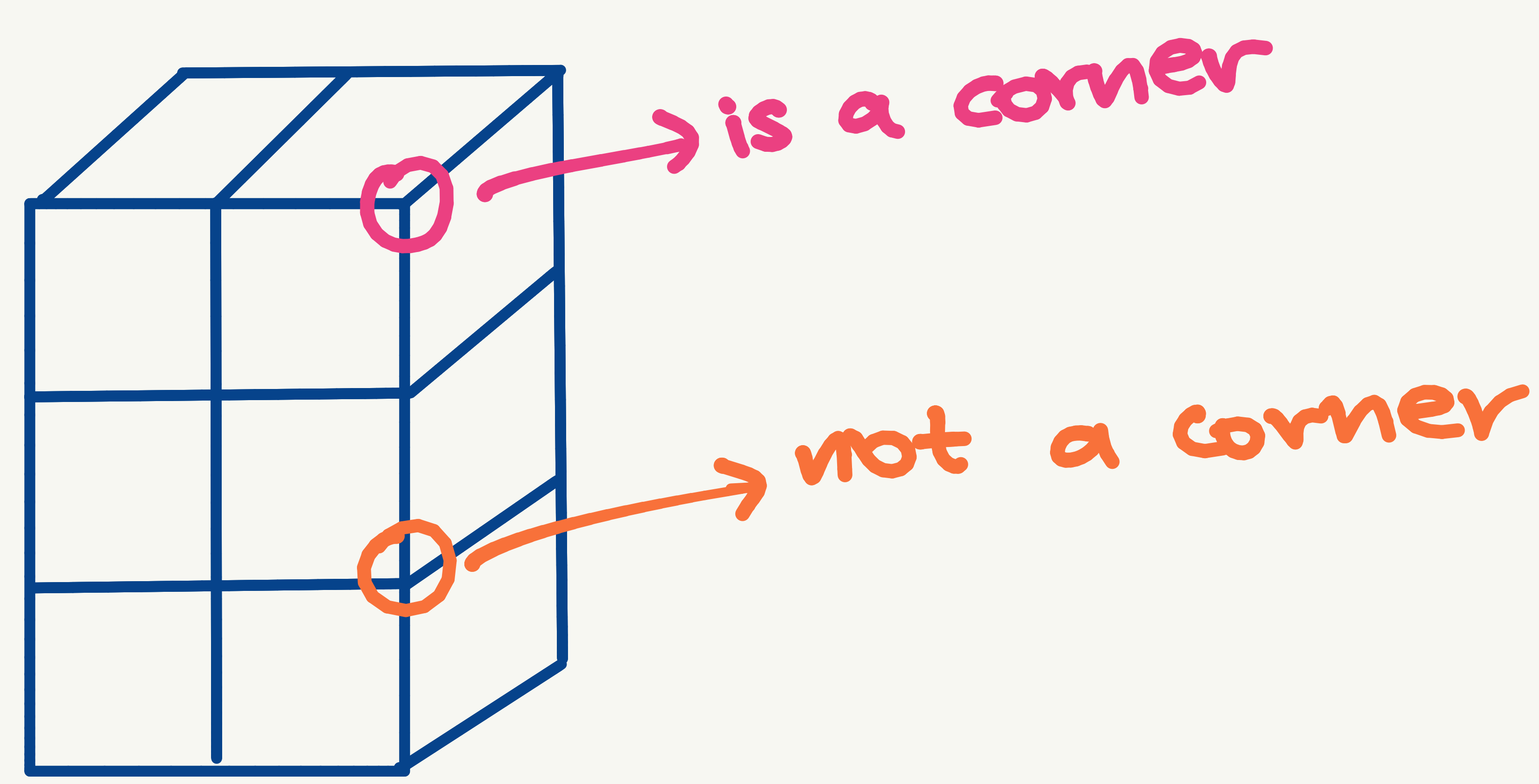
Proof ingredients:

- (Chalopin, Chepoi, Moran, Warmuth 2022) "corner peelings"
  For any finite median graph with basepoint z, any ordering $v_1, v_2, \dots, v_n$ of $V(G)$ with $d(z,v_1) \leq d(z,v_2) \leq \dots \leq d(z,v_n)$ has the property that each v_i is a corner of $G[v_1, \dots, v_i]$



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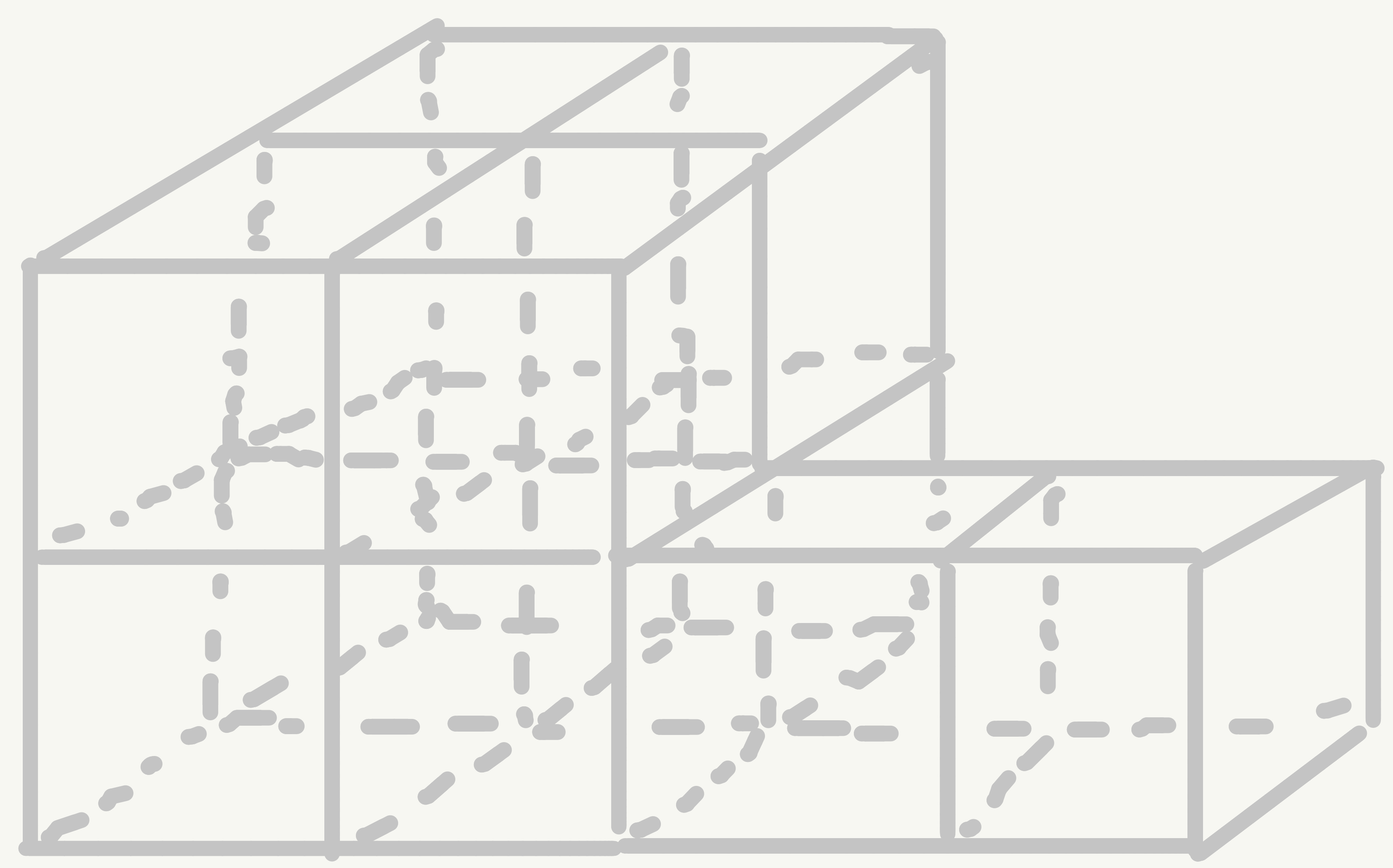

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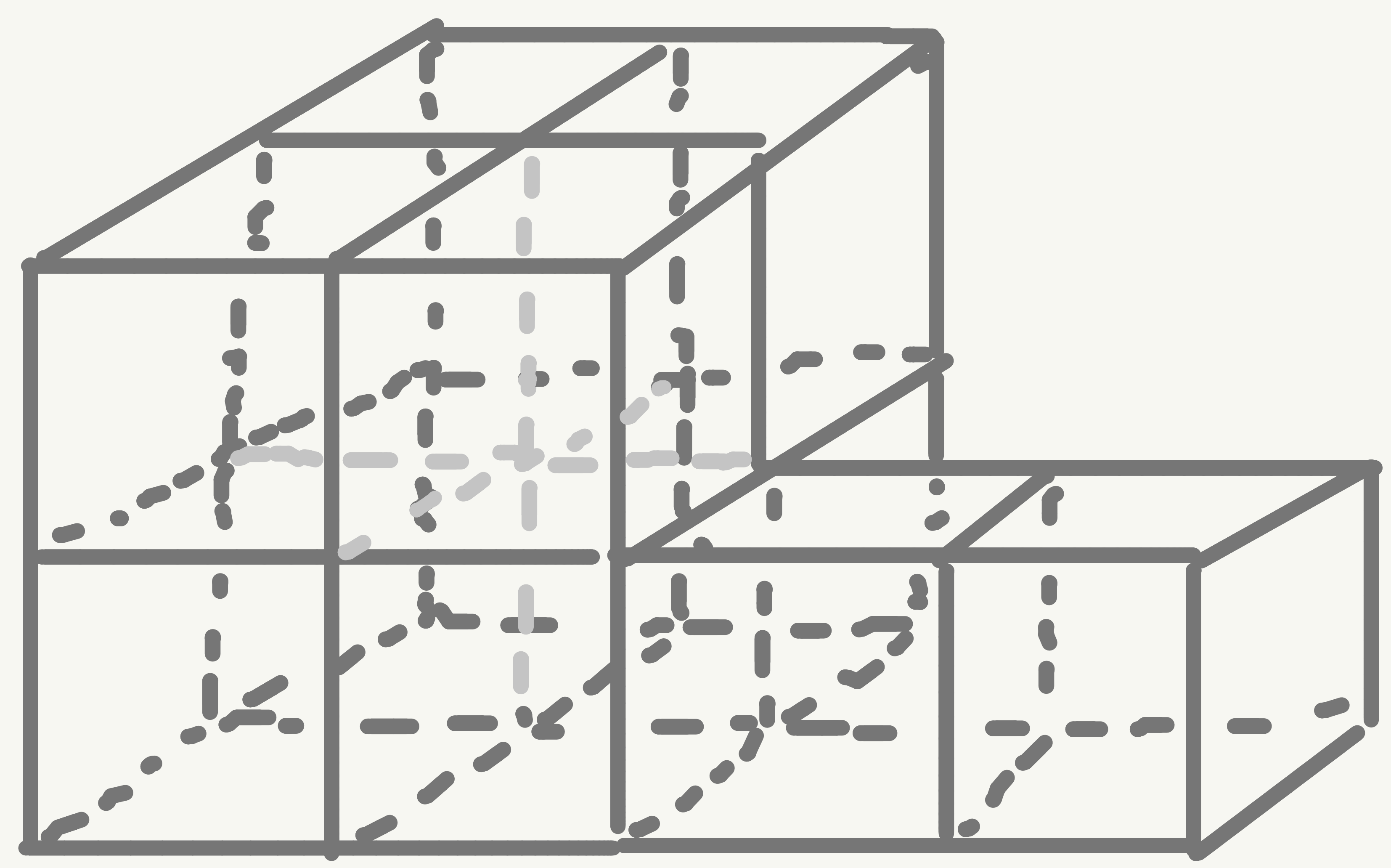

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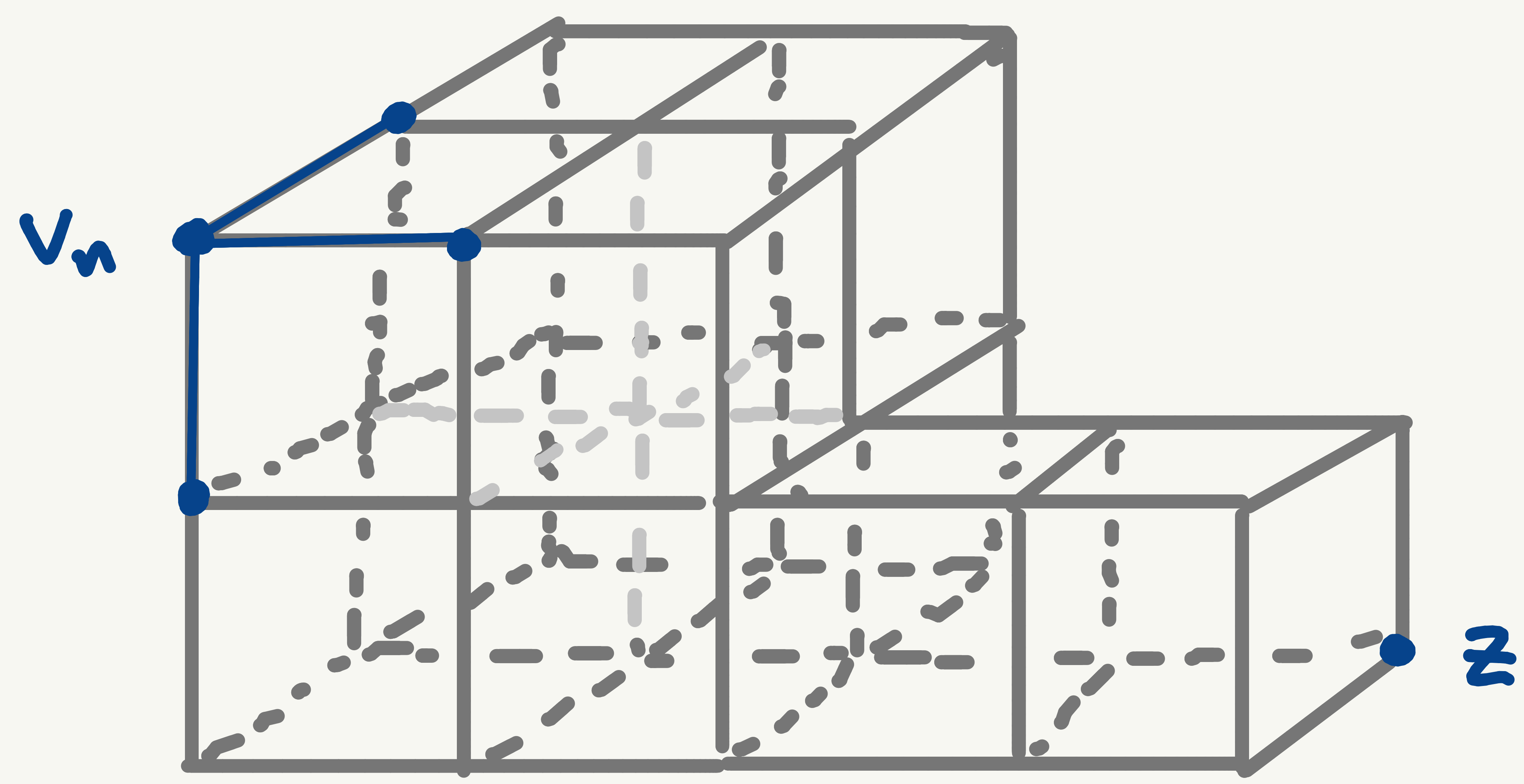

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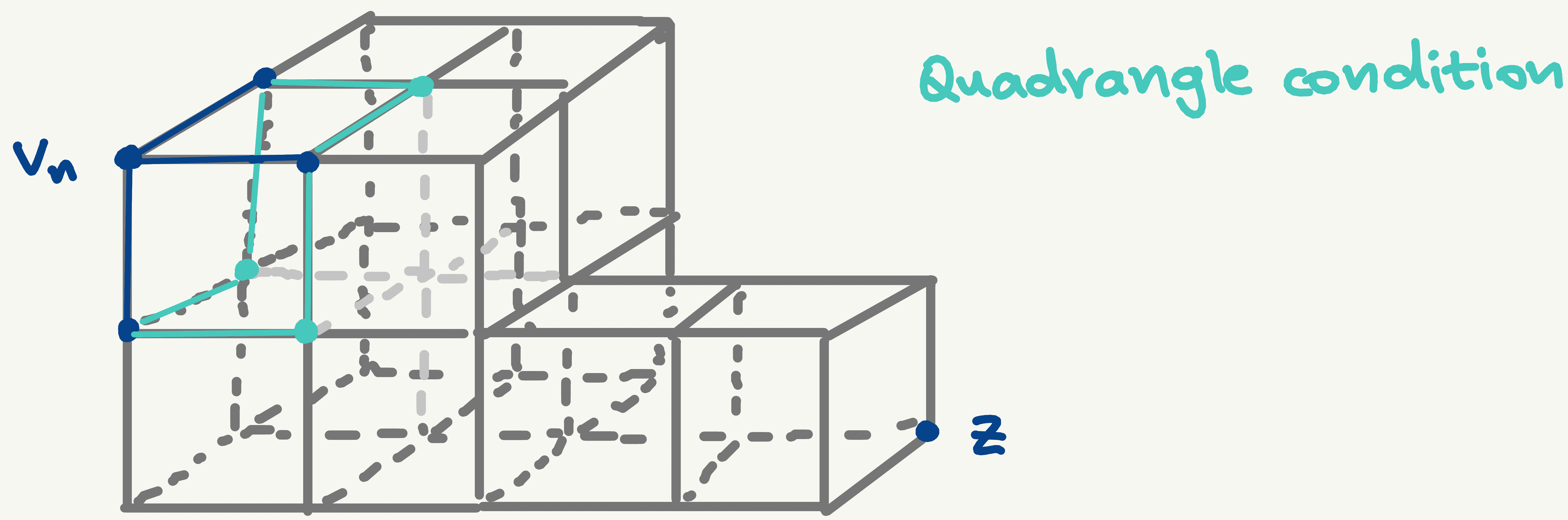

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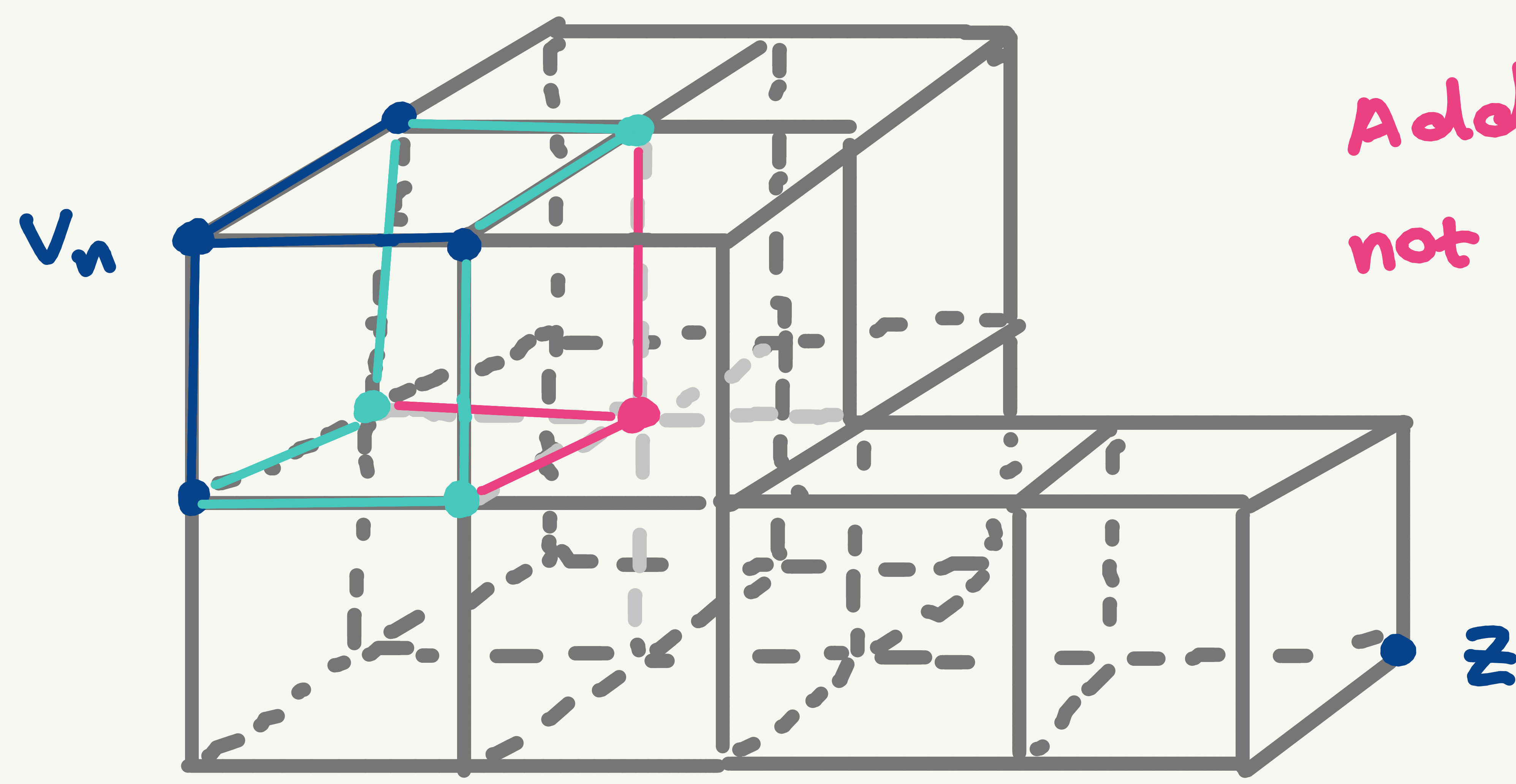

Add opposite vertex if not yet identified.

Obtain distances by comparison with • neighbours.

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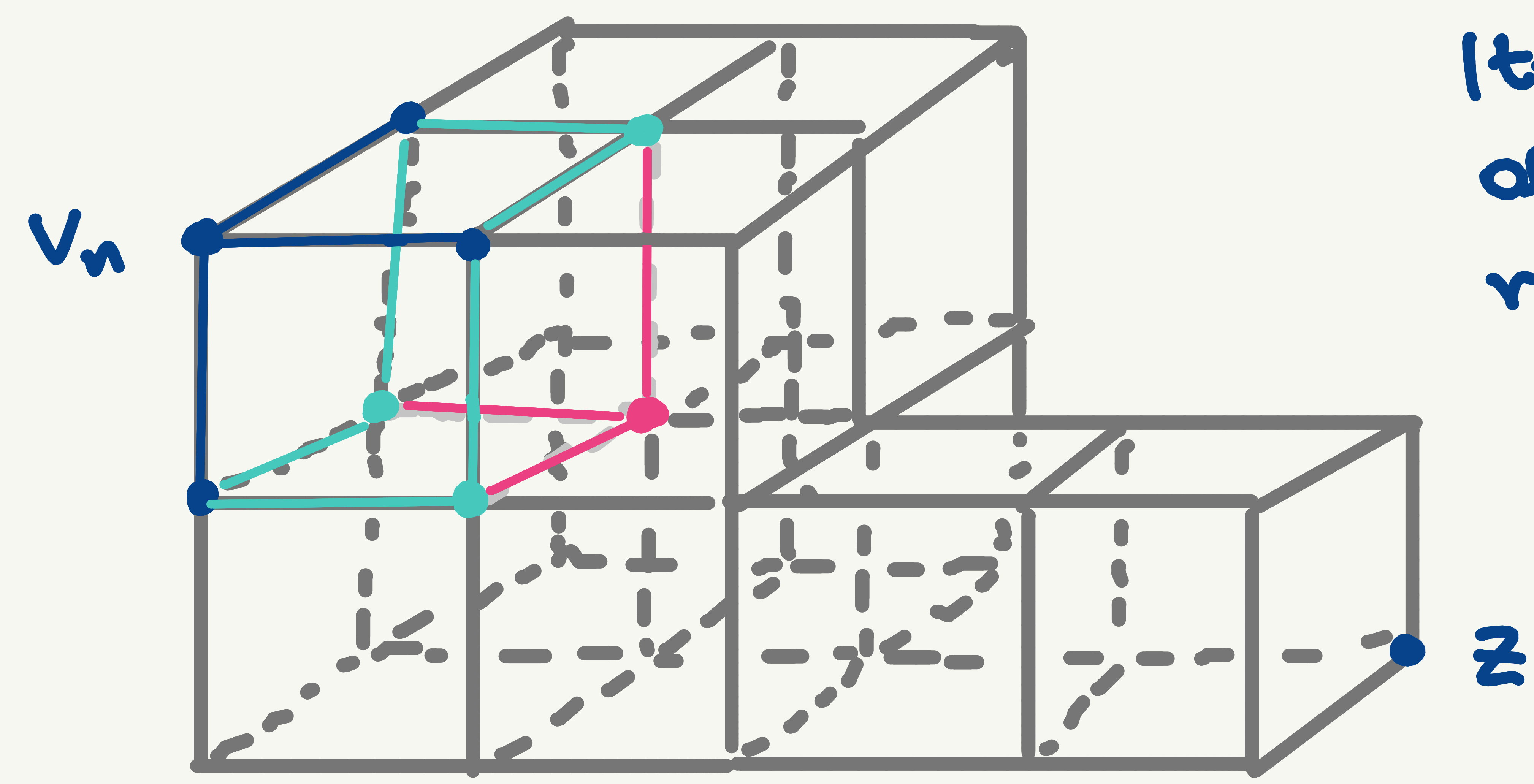

Iterate down ordering of vertices until we reach $v_1 = z$.

Further directions

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Under what conditions
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Realisation questions

e.g. Given a matrix of pairwise distances, can we efficiently decide whether there is a CAT(0) cube complex realising them?

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What other notions from GGT / topology support reconstruction results?

Could such results have any GGT applications?