How to make graph reconstruction harder

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LSE PhD Seminar, 19th November 2021

Classical reconstruction

Definition. *The deck $\mathcal{D}(G)$ of a graph G is the multiset of vertex-deleted subgraphs of G. i.e. $\mathcal{D}(G) = \{G - v : v \in V(G)\}$ (multiset!)*

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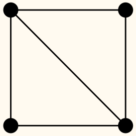

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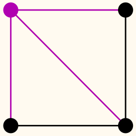

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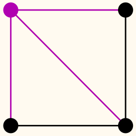

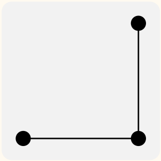

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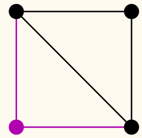

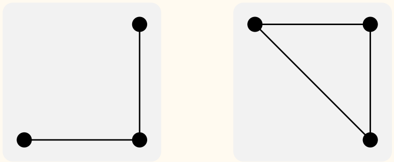

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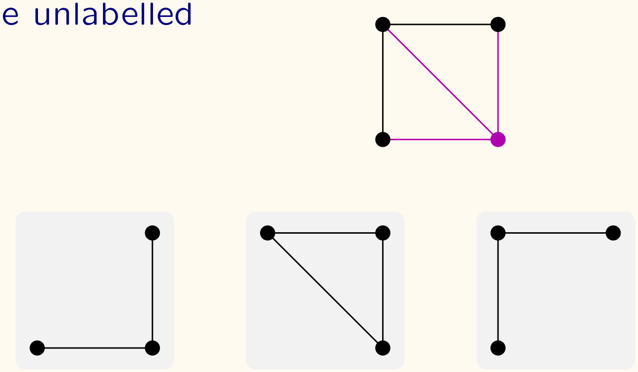

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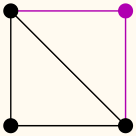

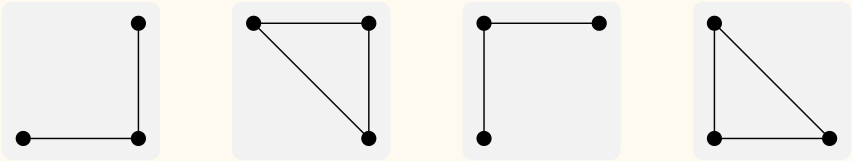

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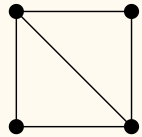

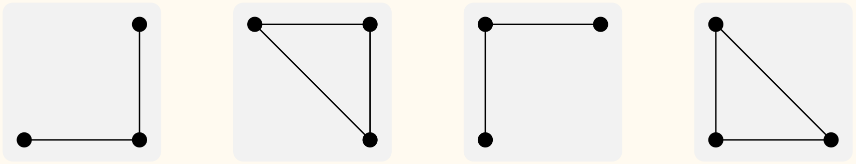

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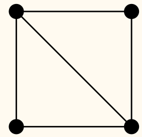

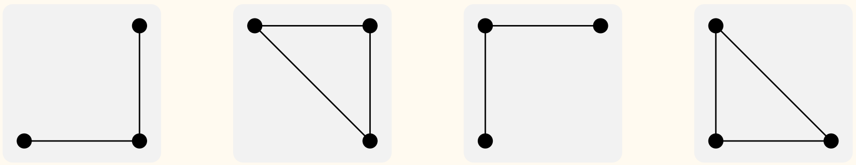

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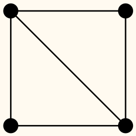

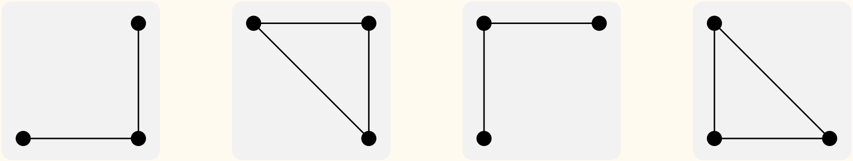

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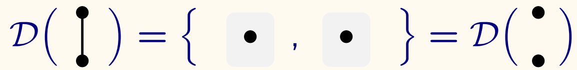

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There are many partial results reconstructing parameters and classes such as:

- # vertices
- # edges, degree sequence, regular graphs, subgraph counts, connectedness, disconnected graphs, trees (Kelly 1942, 1957)
- connectivity, unicyclic graphs (Manvel 1969, 1976)
- Tutte poly, chromatic poly, characteristic poly (Tutte 1967, 1979)
- outerplanar graphs (Giles 1974) maximal planar graphs (Fiorini, Lauri 1981)
- planarity (Bilinski, Kwon, Yu 2006)

Also verified for graphs on up to 13 vertices (McKay 2021).

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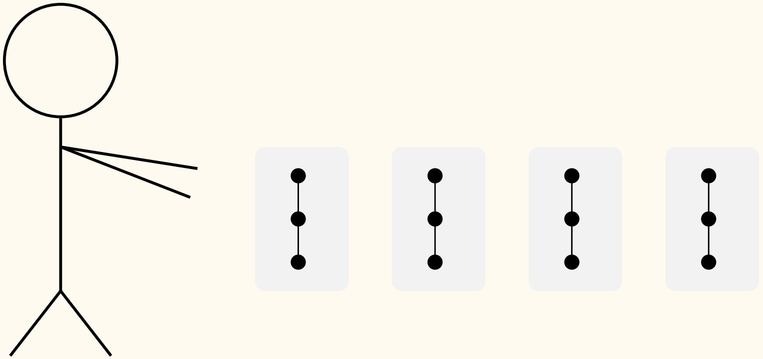

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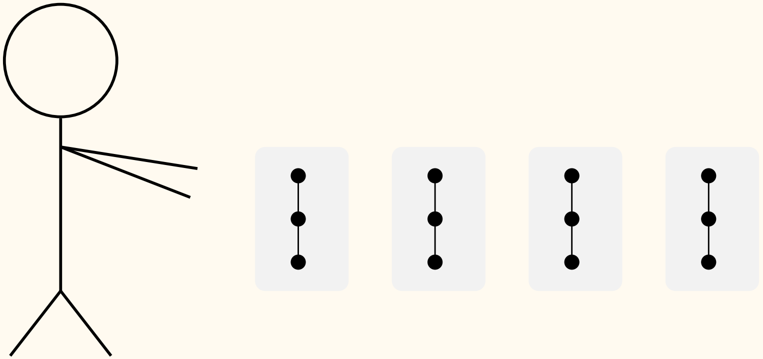

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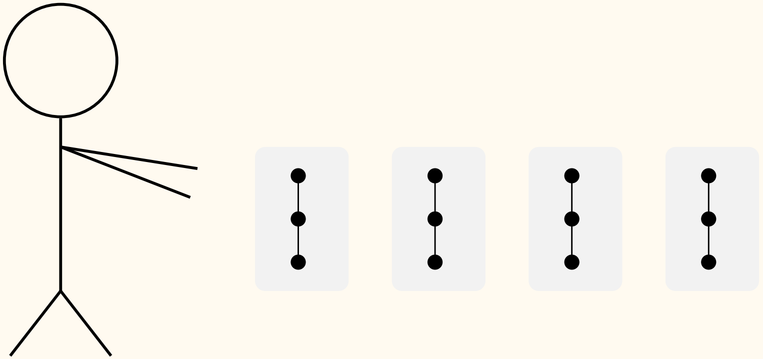

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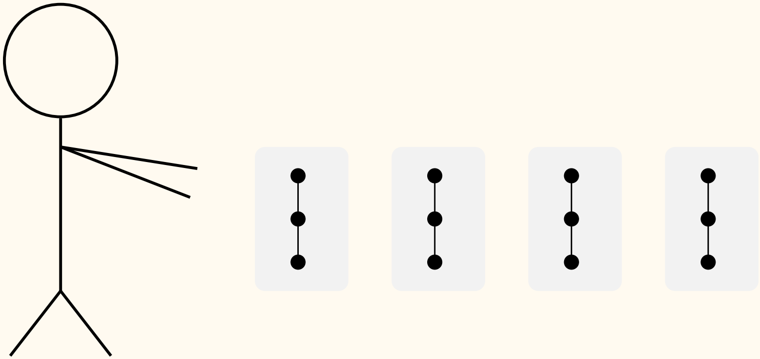

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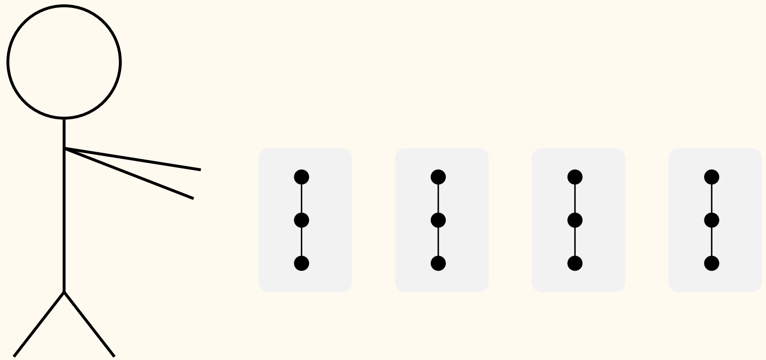

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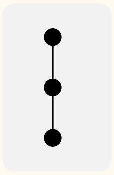

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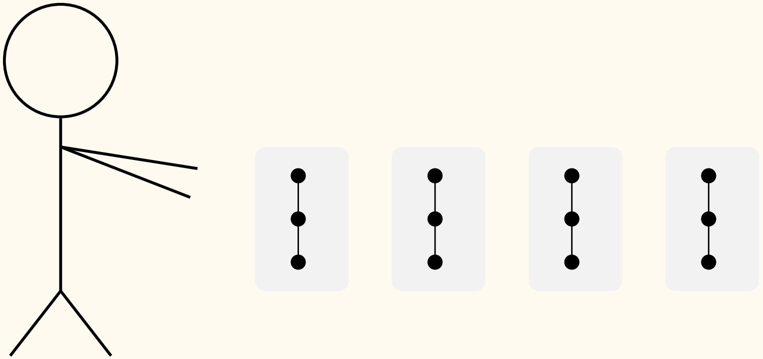

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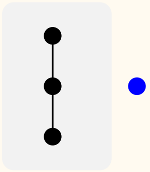

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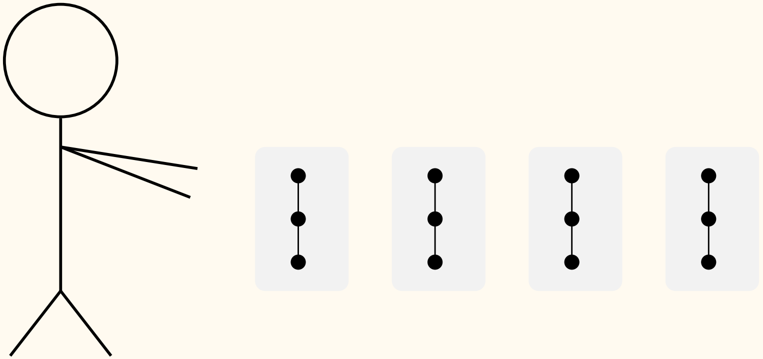

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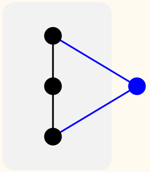

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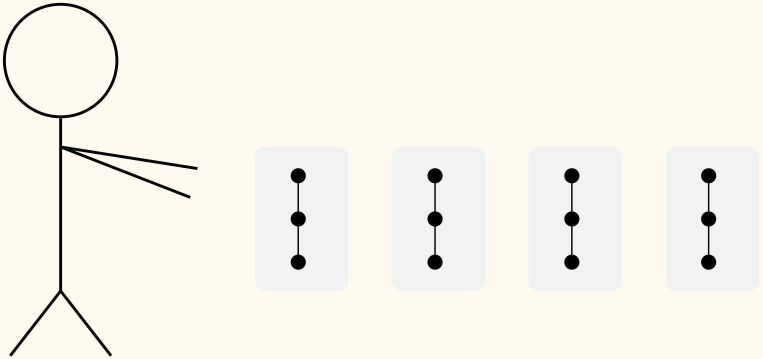

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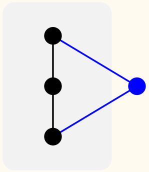

So $G \cong C_4$!

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- Switching reconstruction
  - Card: pick a vertex, switch neighbours and non-neighbours
  - Graphs on n vertices with $n \not\equiv 0 \pmod 4$ are switching-reconstructible
  - (Stanley 1985) Every graph with at least 5 vertices is switching-reconstructible

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Q: For a parameter or class that is reconstructible from $\mathcal{D}(G)$, what is the smallest ℓ for which it is also reconstructible from $\mathcal{D}_\ell(G)$?

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Theorem (Borwein, Erdélyi and Kós)**.** *Suppose that the complex polynomial*

$$p(z) := \sum_{j=0}^{n} a_j z^j$$

has k positive real roots. Then

$$k^2 \leq 2n\log\left(\frac{|a_0| + |a_1| + \cdots + |a_n|}{\sqrt{|a_0 a_n|}}\right).$$

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$$\binom{\alpha_1}{j} + \cdots + \binom{\alpha_m}{j} = \binom{\beta_1}{j} + \cdots + \binom{\beta_m}{j} \quad \text{for all } j \in \{0, \ldots, r\}, \tag{1}$$

then $r + 1 \leq \sqrt{2n\log(2m)}$.

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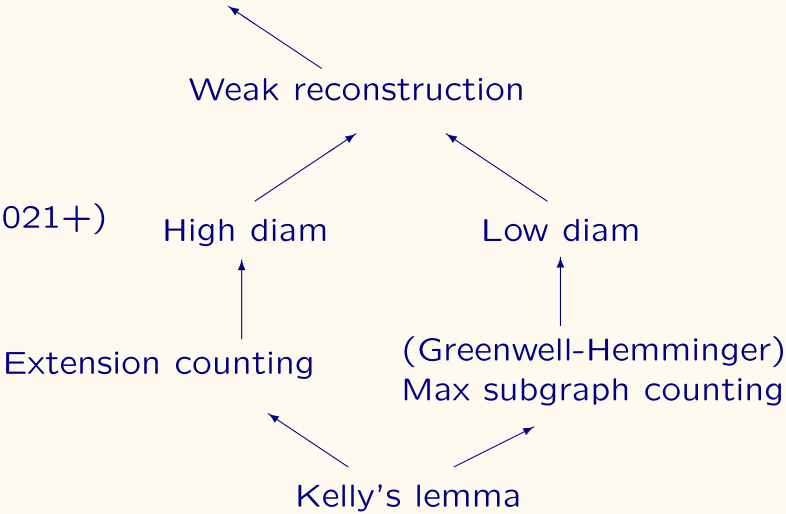

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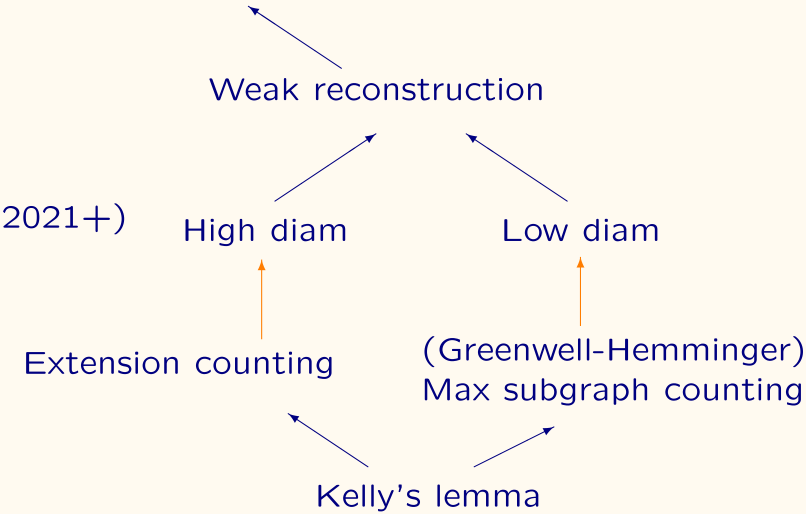

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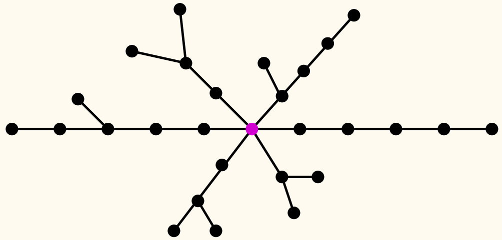

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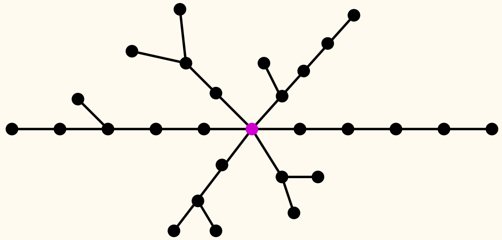

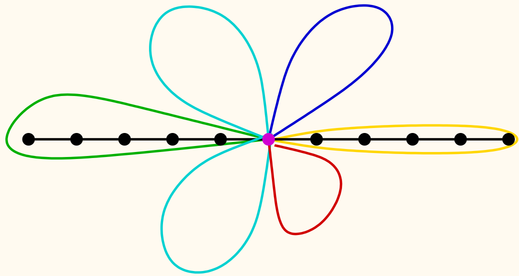

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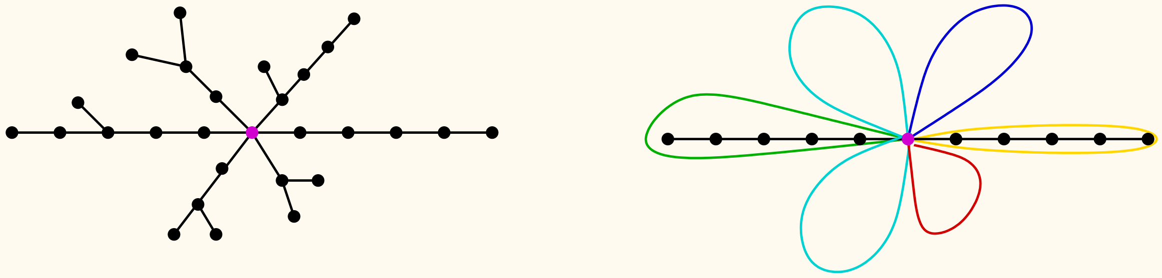
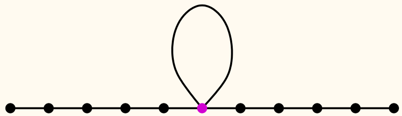
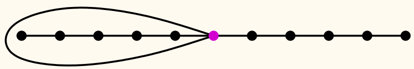


If there is only one P_{k+1}, we can reconstruct branches off the centre by counting maximal copies of

 and 

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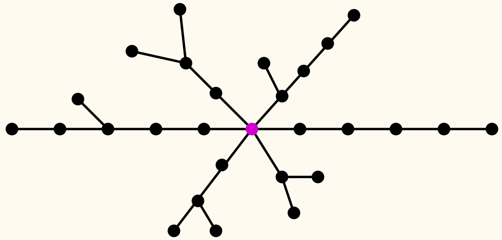

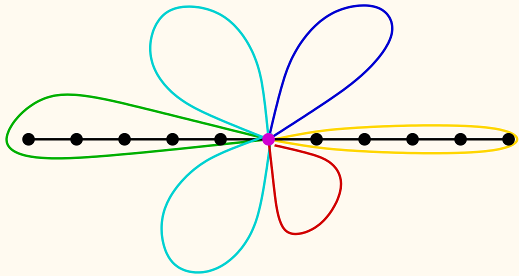

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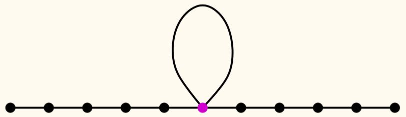

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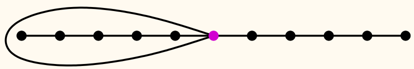

Essence of Lemma (Greenwell-Hemminger)**.** *We can reconstruct the number of maximal copies of these two subgraph types from $D_\ell(G)$ provided the whole subgraph is small enough to be seen on a single card + unique extension condition.*

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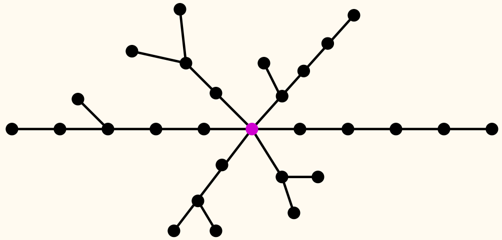
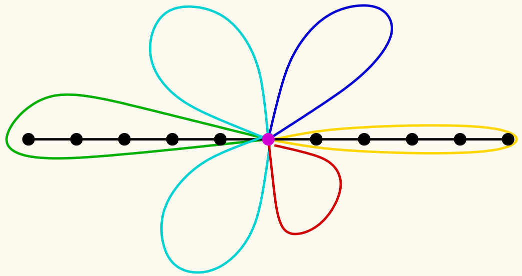
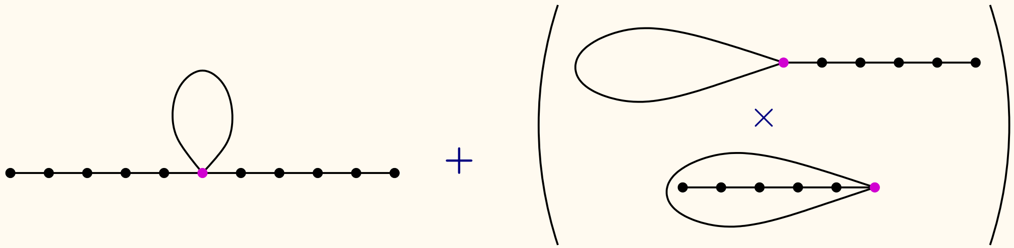




For every possible rooted tree B, we count the number of branches isomorphic to B once per longest path.

i.e. (# branches at c isomorphic to B) $\times$ (# P_{k+1} in T) which can be found by:



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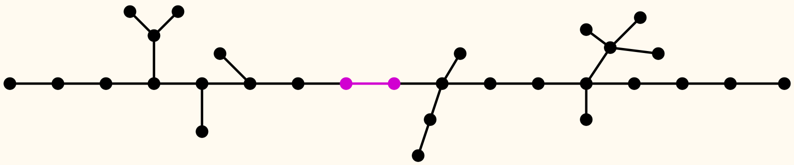

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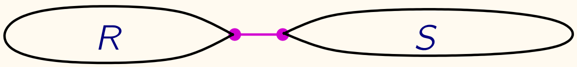

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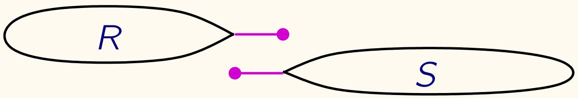

Aside: extension counting

Definition. *Given a graph H, a H-extension is a pair $H_e = (H^+, A)$ where H^+ is a graph and $A \subseteq V(H^+)$ is a subset of vertices with $H^+[A] \cong H$.*

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Lemma. *Let $\ell, d \in \mathbb{N}$ and let G be a graph on at least $\ell + 1$ vertices. For any graph H on at most $\ell - 1$ vertices, at least one of the following conditions must hold:*

1. *There is a copy of H in G whose d-ball in G has at least ℓ vertices.*
2. *For any H-extension H_e, we can reconstruct $m_d(H_e, G)$ from the ℓ-deck of G.*

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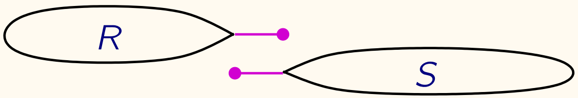

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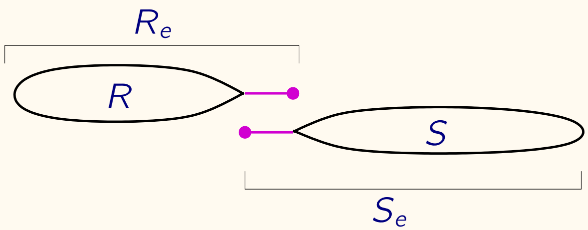

A good pair of
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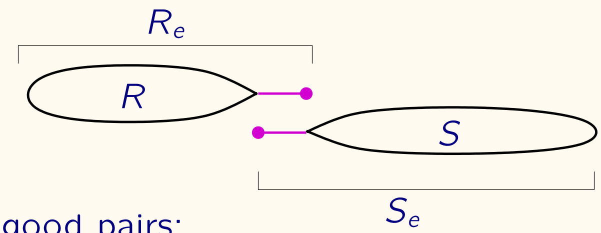

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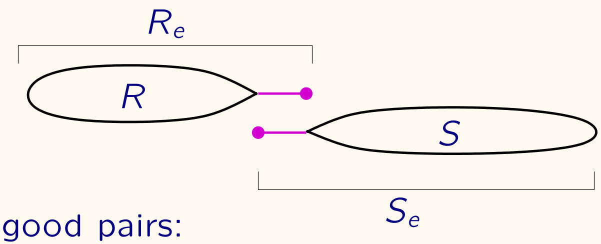

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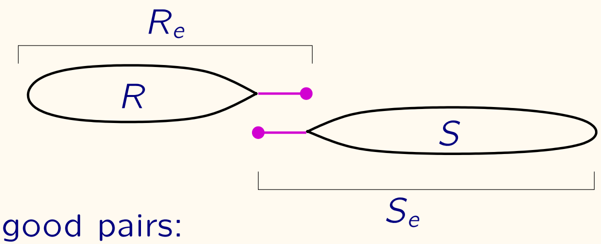

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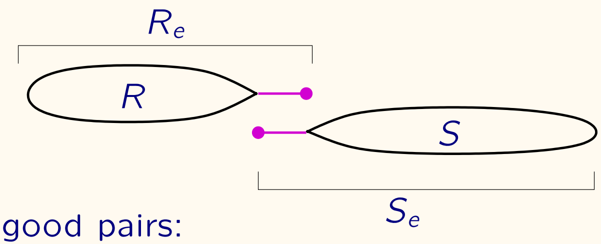

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Essence of Lemma (Extension-counting). *We can count subtrees R whose 1-nbhd has exactly one extra edge and vertex (i.e. R_e) provided all nbhds of copies of R are small enough to fit on a single card.*

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Q: Can we reconstruct the degree sequence from $n-2$ cards in general?